Interior point methods. Quadratic programming

- Convergence of Long-Step Path-Following algorithm
- Quadratic programming
- Interior point method for Quadratic programming

Observations:
If $\sum x_i s_i \geq 0$ then

$$\sqrt{\sum (x_i s_i)^2} \leq \frac{1}{2\sqrt{2}} \sum (x_i + s_i)^2,$$

because

$$\sum_{x_i s_i \geq 0} |x_i s_i| \geq \sum_{x_i s_i \leq 0} |x_i s_i|$$

therefore

$$\sum (x_i s_i)^2 = \sum_{x_i s_i \geq 0} (x_i s_i)^2 + \sum_{x_i s_i \leq 0} (x_i s_i)^2$$

$$\leq \left( \sum_{x_i s_i \geq 0} |x_i s_i| \right)^2 + \left( \sum_{x_i s_i \leq 0} |x_i s_i| \right)^2 \leq 2 \left( \sum_{x_i s_i \geq 0} x_i s_i \right)^2$$

$$\leq 2 \left( \sum_{x_i s_i \geq 0} \frac{1}{4} (x_i + s_i)^2 \right)^2.$$ 

If $x_i s_i \geq \gamma \mu$, $\mu = \frac{1}{n} \sum x_i s_i$

then

$$\sqrt{\sum (x_i s_i)^2} \leq \frac{1}{2\sqrt{2}} (1 + \frac{1}{\gamma}) n \mu$$

Because $\sum \Delta x_i, \Delta s_i = 0$ (homework), hence we can apply the previous observation

$$\sqrt{\frac{|s_i|}{|x_i|}} \Delta x_i + \sqrt{\frac{|x_i|}{|s_i|}} \Delta s_i = \frac{1}{|x_i s_i|} (-x_i s_i + \sigma \mu)$$

Therefore

$$\sqrt{\sum |x_i s_i|^2} \leq \frac{1}{2\sqrt{2}} \sum \left( x_i s_i - 2\sigma \mu + \sigma^2 \mu^2 \frac{n}{\gamma \mu} \right) \leq \frac{1}{2\sqrt{2}} (1 + \frac{1}{\gamma}) n \mu.$$ 

**Theorem** For fixed $\gamma, \sigma_{\min}, \sigma_{\max}$ there is $\delta > 0$

$$\mu_{k+1} \leq (1 - \frac{\delta}{n}) \mu_k.$$ 

**Proof** We first want to check how long the iteration step $\alpha_k$ can be taken, that is

$$(x_i + \alpha_k \Delta x_i)(s_i + \alpha_k s_i) \geq \gamma \frac{1}{n} \sum (x_i + \alpha_k \Delta x_i)(s_i + \alpha_k s_i).$$
This certainly holds (see book) if
\[ \alpha_k \leq 2\sqrt{\frac{\sigma_k}{n}} \frac{1 - \gamma}{1 + \gamma}. \]

Hence we can at least take a step of length
\[ \alpha_k = 2\sqrt{\frac{\sigma_k}{n}} \frac{1 - \gamma}{1 + \gamma}. \]

Therefore
\[ \mu_{k+1} = \frac{1}{n} \sum_i (x_i + \alpha_k \Delta x_i)(s_i + \alpha_k s_i) = \mu_k(1 - \alpha_k(1 - \sigma_k)) \leq \mu_k \left(1 - \frac{1}{n} 2\sqrt{2}\sigma_k(1 - \sigma_k) \frac{1 - \gamma}{1 + \gamma}\right). \]

Since \( \sigma_k \) is bounded from above and from below, so is \( \sigma_k(1 - \sigma_k) \), and we are done.

**Theorem** The algorithm converges in \( O(n) \) iterations: if \( \mu_0 \leq 1/\epsilon \), then for \( K = O(n \log 1/\epsilon) \)
\[ \mu_K \leq \epsilon. \]

**Proof** Take logs, then
\[ \log \mu_k \leq k \log (1 - \delta/n) + \log \mu_0 \approx \log \mu_0 - \frac{k\delta}{n}. \]

**Quadratic optimization** In general
\[ \min_x \frac{1}{2} \langle x, Qx \rangle + \langle c, x \rangle, \ Ax = b, \ x \geq 0, \ x \in \mathbb{R}^n, \ b \in \mathbb{R}^m. \]

Lagrangian
\[ \mathcal{L} = q(x), \ q(x) = \frac{1}{2} \langle x, Qx \rangle + \langle c, x \rangle + \langle \lambda, (Ax - b) \rangle - \langle s, x \rangle. \]

KKT conditions:
\[ Qx + A^t \lambda - s = c, \]
\[ Ax = b, \]
\[ \langle x, s \rangle = 0, \]
\[ x_i \geq 0, s_i \geq 0. \]

Solve for
\[ \begin{pmatrix} Q & A^t & -I \\ A & 0 & 0 \\ \text{diag}(s^k_i) & 0 & \text{diag}(x^k_i) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{pmatrix} = \begin{pmatrix} -Q - A^t \lambda^k + s^k + c \\ -Ax^k + b \\ -\text{diag}(s^k_i x^k_i) \end{pmatrix}. \]

Algorithm is identical to linear interior point methods.
Consider a special case, no constraint \( x_i \geq 0 \).

**Theorem** If \( A \) has a full rank and \( G \) is positive-definite, on the null-space of \( A \), then there is a unique global solution of the quadratic minimization problem

**Proof** In this case the problem deals with the KKT matrix
\[ K = \begin{pmatrix} Q & A^t \\ A & 0 \end{pmatrix}. \]
If $K(x, \lambda)^t = 0$, then $Ax = 0$, $x$ is in the null-space of $A$. But

$$0 = \langle (x, \lambda)^t, K(x, \lambda)^t \rangle = \langle x, Qx \rangle.$$

Hence $x = 0$, which immediately implies that $\lambda = 0$ as well.

Suppose $(x^*, \lambda^*)$ is the solution of the KKT conditions (it exists because by the previous step the KKT matrix is nonsingular), then any other $(x, \lambda)$ satisfies

$$q(x) = q(x^*) + \frac{1}{2} \langle (x - x^*), Q(x - x^*) \rangle.$$

Remark If $Q$ is positive-definite, then the statement, obviously holds. Moreover, it can be extended to convex constrained minimization problems.