Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2009,**29B**(4):777–802
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

TWO-DIMENSIONAL RIEMANN PROBLEMS: FROM SCALAR CONSERVATION LAWS TO COMPRESSIBLE EULER EQUATIONS*

Dedicated to Professor Wu Wenjun on the occasion of his 90th birthday

Li Jiequan (李杰权)
School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
E-mail: jiequan@mail.cnu.edu.cn

Sheng Wancheng (盛万成)
Department of Mathematics, Shanghai University, Shanghai 200444, China
E-mail: mathwcsheng@shu.edu.cn

Zhang Tong (张同)
Institute of Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, China
E-mail: tzhang@math.ac.cn

Zheng Yuxi (郑玉玺)
Department of Mathematics, The Pennsylvania State University, UP, PA 16802, USA
E-mail: yzheng@math.psu.edu

Abstract In this paper we survey the authors' and related work on two-dimensional Riemann problems for hyperbolic conservation laws, mainly those related to the compressible Euler equations in gas dynamics. It contains four sections: 1. Historical review. 2. Scalar conservation laws. 3. Euler equations. 4. Simplified models.

Key words two-dimensional Riemann problem; compressible Euler equation; reflection of shocks; interaction of rarefaction waves; delta-shocks

2000 MR Subject Classification 35L65; 35L67; 76J20; 76N15

1 Historical Review

In 1860, B. Riemann [23, 63] proposed and investigated the initial-value problem with the simplest discontinuous initial data for isentropic Euler equations:

*Received December 29, 2008. The research of Li partially supported by 973 Key program and the Key Program from Beijing Educational Commission with No. KZ200910028002, Program for New Century Excellent Talents in University (NCET) and Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR-IHLB); The research of Sheng partially supported by NSFC (10671120) and Shanghai Leading Academic Discipline Project: J50101; The research of Zhang partially supported by NSFC (10671120); The research of Zheng partially supported by NSF-DMS-0603859

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{(conservation of mass)} \\ (\rho u)_t + (\rho u^2 + p)_x = 0, & \text{(conservation of momentum)} \\ (\rho, u)(x, 0) = (\rho_\pm, u_\pm), \quad \pm x > 0. & \end{cases} \tag{1.1}$$

where ρ, p and u are the density, pressure and velocity, respectively, $p = A\rho^\gamma$, and $\gamma > 1$ is the adiabatic index. The initial data consist of two arbitrary constant states $\ominus := (\rho_-, u_-)$ and $\oplus := (\rho_+, u_+)$. It is natural to seek self-similar solutions, i.e., $(u, \rho)(x, t) = (u, \rho)(\xi)$, $\xi = \frac{x}{t}$, due to the fact that both the equations and the initial data are self-similar.

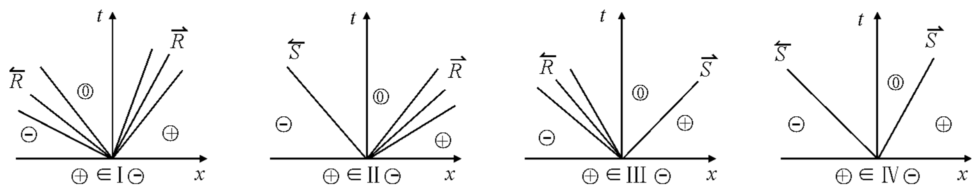


Figure 1.1

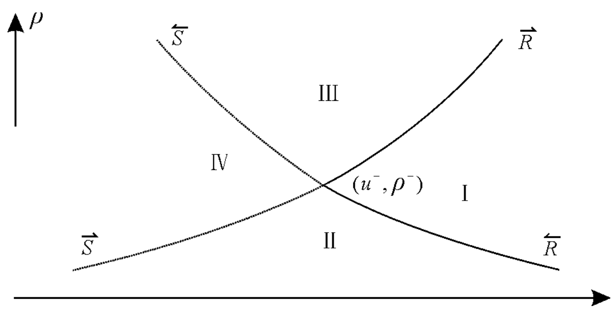


Figure 1.2

He initiated the concept of weak solutions and the method of phase plane analysis, constructed the four types of solutions, and clarified their criteria as shown in Figures 1.1–1.2. Where

$$\overset{\leftharpoondown}{\underset{\rightharpoonup}{R}} : \begin{cases} \xi = \lambda \mp := u \mp (p'(\rho))^{\frac{1}{2}}, \\ \mathrm{d}u = \mp \dfrac{(p'(\rho))^{\frac{1}{2}}}{\rho} \mathrm{d}\rho, \end{cases}$$

$$\overset{\leftharpoondown}{\underset{\rightharpoonup}{S}} : \begin{cases} \omega = u_0 \mp \Big(\dfrac{\rho}{\rho_0} \dfrac{[p]}{[\rho]} \Big)^{\frac{1}{2}}, \\ \Big(\dfrac{1}{\rho\rho_0} \dfrac{[p]}{[\rho]} \Big)^{\frac{1}{2}} [\rho] = [u], \end{cases}$$

$$p\big|_{\text{wave front}} < p\big|_{\text{wave back}},$$

$u := u(\omega + 0)$, $u_0 := u(\omega - 0)$, $[u] := u(\omega + 0) - u(\omega - 0)$, $\lambda\mp$ are eigenvalues of system (1.1), $\overleftarrow{R}(\overrightarrow{R})$ is backward (forward) rarefaction wave, and $\overleftarrow{S}(\overrightarrow{S})$ is backward (forward) shocks, ω is the propagating speed of shock.

In 1957, P. Lax contributed the well-known theory:

Theorem 1.1 Consider general one-dimensional hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

where $u = (u_1, \cdots, u_m)$ and $f = (f_1, \cdots, f_m)$, with Riemann initial data

$$u(x, 0) = u_\pm, \quad \pm x > 0. \tag{1.3}$$

Assume strict hyperbolicity of the system and genuine nonlinearity (convex) or linear degeneracy of each characteristic field and $|u_+ - u_-| \ll 1$. Then Riemann problem (1.2) and (1.3) has a solution, which consists of $m+1$ constant states connected by centered waves (rarefaction waves, shocks or contact discontinuities).

Lax's theorem not only shows the existence of solutions, but also provides a very clear picture of solution structures. Moreover, Lax in that paper clarified many important concepts: Genuine nonlinearity, linear degeneracy, entropy condition, etc.

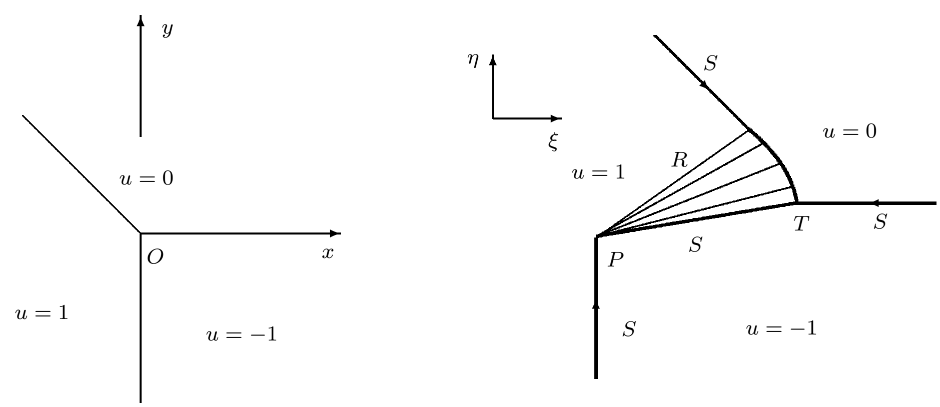
In 1959, Gelfand [31] states "the Riemann problem plays a special role in the theory of quasilinear hyperbolic systems. As we will show, the Riemann problem are important in the study of existence, uniqueness and asymptotic behavior of solutions as $t \to \infty$ for the corresponding Cauchy problem. Besides, the study of the Riemann problem has its own sake."

It is well recognized nowadays that the Riemann problem plays the role of "building blocks" for all fields of theory, numerics and applications. "It has so far been the most important problem in the entire field of conservation laws."*

2 Scalar Conservation Laws

The study of two-dimensional Riemann problems of scalar conservation laws was dated back to Guckenheimer in [35]. In 1975, he constructed a Riemann solution for

$$u_t + \left(\frac{u^2}{2}\right)_x + \left(\frac{u^3}{3}\right)_y = 0 \tag{2.1}$$



(a) Gukenheimer initial data

(b) Gukenheimer solution

Figure 2.1 The solution structure for the Gukenkeimer problem

with the initial data shown in Figure 2.1 (a). The solution structure is displayed in Figure 2.1 (b). Interestingly, we note that a planar shock bifurcates at point P into a composite wave (a centered simple wave adjacent to a shock). This is a typical, genuinely two-dimensional wave

*Book Reviews in Bulletin of the American Mathematical Society, 1991, **24**(1): 228–234 phenomenon, which never shows up in the one-dimensional case. Furthermore, it resembles the Mach reflection configuration in the neighborhood of a triple point T (cf. Figure 2.3 (a_2)).

The systematic study of the two-dimensional Riemann problem for

$$u_t + f(u)_x + g(u)_y = 0, \quad u \in \mathbb{R}, \tag{2.2}$$

with the initial data

$$u|_{t=0} = u_i := ⓘ, \qquad (x,y) \in i\,\text{th quadrant}, \qquad i = 1,2,3,4, \tag{2.3}$$

where $u_i (1 \le i \le 4)$ are constant states were from [76, 83]. In [76], Wagner constructed solution of this problem under the assumptions $f''(u) > 0$, $g''(u) > 0$ and f is almost the same as g. The last assumption means that the problem is essentially one-dimensional, in the sense that one can use one-dimensional methods to investigate it. In [83], instead of the third assumption, Zhang and Zheng found the assumption $(f''(u)/g''(u))' > 0$ which means that the problem is the simplest and genuinely two-dimensional, and constructed the solutions by using the method of generalized characteristic analysis* (the analysis of characteristics, shocks, sonic curves and the law of causality) designed by themselves. For self-similar solutions, the problem reduces to

$$(f'(u) - \xi)u_\xi + (g'(u) - \eta)u_\eta = 0, \tag{2.4}$$

where $\xi = \frac{x}{t}, \eta = \frac{y}{t}$ and the initial data become boundary data at infinity. The characteristic equation of (2.4) takes the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\eta - g'(u)}{\xi - f'(u)}, \qquad \frac{\mathrm{d}u}{\mathrm{d}\xi} = 0. \tag{2.5}$$

Integrating this system, we obtain the characteristic lines

$$\frac{\eta - g'(u)}{\xi - f'(u)} = C_1, \qquad u = C_2, \tag{2.6}$$

where C_1 and C_2 are arbitrary constants. Obviously the characteristic lines of constant solution u_0 are all straight lines coming from infinity and focusing at the singularity point $(f'(u_0), g'(u_0))$. All shocks jumping from constant states u_1 to u_2 must be straight lines and focus at the singularity point $(f'_{1,2}, g'_{1,2})$, where $f'_{1,2} := \frac{f(u_1)-f(u_2)}{u_1-u_2}$, $g'_{1,2} := \frac{g(u_1)-g(u_2)}{u_1-u_2}$.

Now we are at the stage to construct the Riemann solution. The boundary data at infinity have four jumps, each jump emits a one-dimensional wave: R or S. According to the combinations of R and S, the Riemann problem can be divided into five cases: $4R$, $4S$, $2R+2S$, $3R+S$ and $3S+R$. The waves go forward and match together at their singularity points eventually. The detailed progress is shown in the following figures. In Figure 2.2 (a), $4R$s match at their singularity points directly. In Figure 2.2 (b), both $S_{1,2}$ $S_{2,3}$ and $S_{3,4}$ $S_{1,4}$ intersects, respectively, at two points and form two $S_{1,3}$, then two $S_{1,3}$ match together at the singularity point $(f'_{1,3}, g'_{1,3})$. In Figure 2.2 (c), $S_{1,4}$ penetrates $R_{3,4}$ and changes to $S_{1,3}$. $S_{2,3}$ passes through $R_{1,2}$ and its strength is weakened to zero. It is interesting that a non-planar rarefaction wave appears at the right of the shock and the envelope of its characteristics is just the shock. The rest part

*The name of this method did not appear until a later time in [82] includes that the planar rarefaction wave and non-planar rarefaction wave can match together continuously at their singularity points and $S_{1,3}$ can pass through the non-planar rarefaction wave till its strength becomes zero somewhere.

After that, Zhang and Zhang [82] developed the generalized characteristic analysis method and obtained the necessary and sufficient condition for the appearance of Guckenheimer structure for (2.2) with initial data consisting of three constant states. Later researches for (2.2) [3, 16, 67, 69] were done almost under such an assumption. Later researches for (2.2) [3, 16] were done almost under such an assumption. Sheng [67] gave the condition

$$f'' > 0, \quad ug'' > 0 (u \neq 0), \quad (g''/f'')' > 0, \tag{2.7}$$

which is satisfied by Guckenheimer's example, and clarified the necessary and sufficient condition for the appearance of Guckenheimer structure for (2.2).

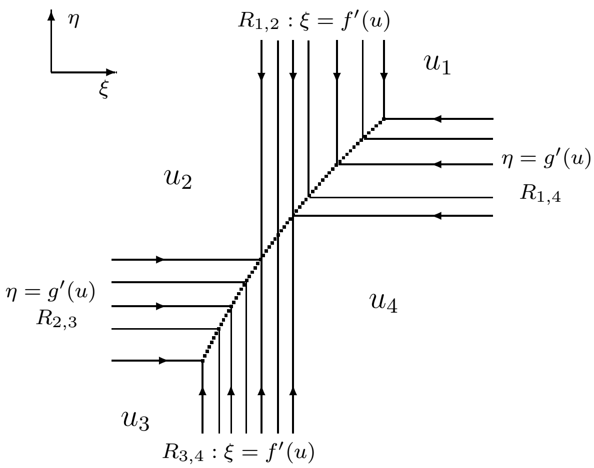


Figure 2.2 (a) Case $u_3 \le u_2 \le u_4 \le u_1$

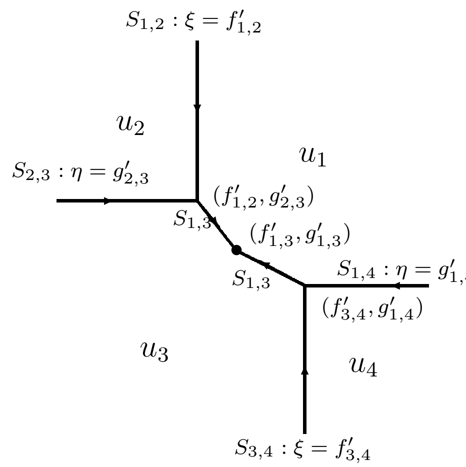


Figure 2.2 (b) Case $u_3 \ge u_4 \ge u_2 \ge u_1$

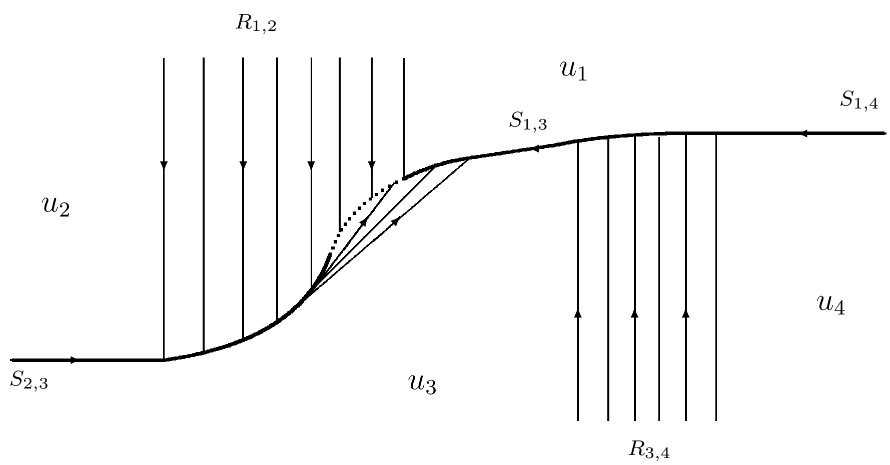


Figure 2.2 (c) Case $u_2 < u_1 < u_3 < u_4$

Perhaps the most famous example of two-dimensional Riemann problems is the reflection of an oblique shock (ramp climbing problem of shock). The reflection of an oblique shock (ramp climbing problem of shock) is a very famous problem which is tight to two-dimensional Riemann problems. This phenomenon was observed experimentally by Mach [59] and it was rejuvenated by von Neumann [60]. Research efforts on it have lasted over one century. According to the physical and/or numerical experiments, this problem exhibits rich patterns of fluid flows [4]: Regular reflection (RR), Mach reflection (MR) and von Neumann reflection (NR) [22, 80] see Figure 2.3$(a_1),(a_2),(a_3)$. In the case of RR, the incident shock I touches the ramp and reflects, while in the case of MR, the incident shock I ends at a triple point T and reflects, and the triple point T connects the ramp through a Mach shock M. In the case of NR, instead of the triple point, there is a continuous wave fan. But the criteria for these three patterns is a mystery so far.

Sheng and Zhang [69] considered the ramp climbing problem of shocks for scalar conservation law (2.2). f and g satisfy $f'_{1,2}>0$ and the condition (2.7). For simplicity, they assumed that

$$f(0)=0,\ g(0)=0; f'(0)=0,\ g'(0)=0. \tag{2.8}$$

The initial-boundary value is

$$u(0,x,y)=u_0(x,y)=\begin{cases} u_1, & 0<x<+\infty, \quad \tan\theta\ x<y<+\infty,\\ u_2, & -\infty<x<0, \quad y>0,\end{cases} \tag{2.9}$$

$$u(t,x,y)|_\Gamma=p(t,x,y):=\begin{cases} u_1, & \text{on } \Gamma_1,\\ u_2, & \text{on } \Gamma_2,\end{cases} \tag{2.10}$$

where $u_1<u_2$ are constants, $\theta\in(0,\frac{\pi}{2})$ is an angle of the ramp,

$$\Omega=\{(t,x,y)|t\ge 0, x\ge 0, y\ge \tan\theta\ x\}\bigcup\{(t,x,y)|t\ge 0, x\le 0, y\ge 0\}$$

with the boundary $\Gamma=\Gamma_1\bigcup\Gamma_2$ and $\Gamma_1=\{(t,x,y)|t\ge 0, 0<x<+\infty, y=\tan\theta\ x\}$, $\Gamma_2=\{(t,x,y)|t\ge 0,-\infty<x<0, y=0\}$.

Sheng and Zhang constructed solutions to problem (2.2) and (2.9)–(2.10) as shown in Figure 2.3 $(b_1)-(b_3)$ and proved the following results.

Theorem 2.1 Let θ be the angle of the ramp,

$$\theta_2=\arctan g'(u_2)/f'(u_2) \ \text{ and } \ \theta_{12=}\arctan(g'_{1,2}/f'_{1,2}).$$

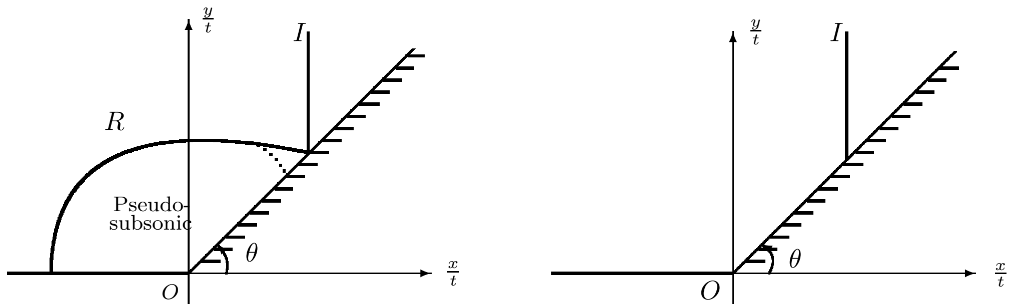
Then the criteria for the three configurations are as follows:

1) The appearance of RR-like configuration $\leftrightarrows \theta\ge\max\{\theta_2,\theta_{12}\}$;

2) The appearance of MR-like configuration $\leftrightarrows \theta_{12}\ge\max\{\theta,\theta_2\}$;

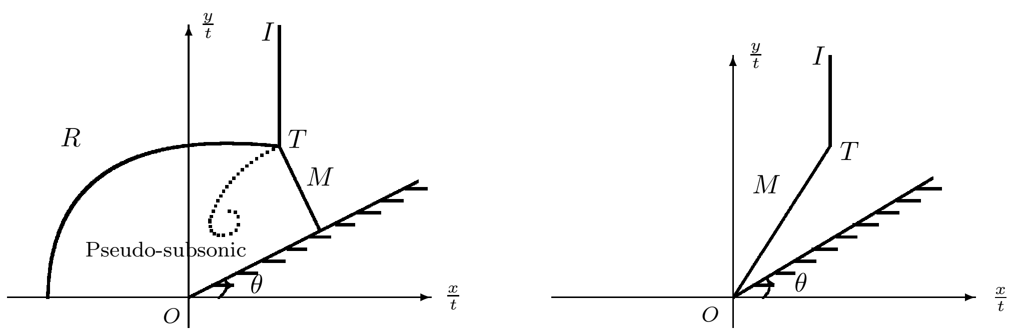
3) The appearance of NR-like configuration $\leftrightarrows \theta_2\ge\max\{\theta_{12},\theta\}$.

Let us compare the $(a_1)-(a_3)$ with $(b_1)-(b_3)$ in Figure 2.3(1)–(3): It is interesting to see that (a)s change to (b)s when the subsonic domains shrink to the origin, since scalar conservation laws have only one real eigenvalues. We may think (b)s are cartoons of (a)s.



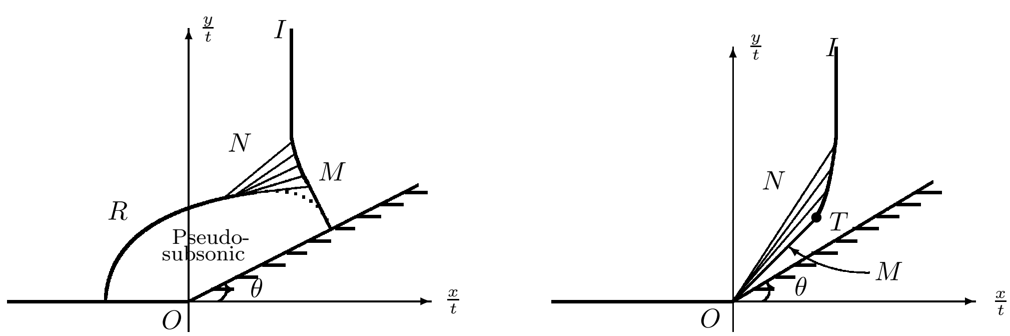
(a_1) RR configuration (b_1) RR-like configuration

Figure 2.3 (1)



(a_2) MR configuration (b_2) MR-like configuration

Figure 2.3 (2)



(a_3) NR configuration (b_3) NR-like configuration

Figure 2.3 (3)

3 Compressible Euler Equations

In 1990, Zhang and Zheng [84] considered the two-dimensional Riemann problem for compressible Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, & \text{(conservation of mass)} \\ \left.\begin{aligned} &(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ &(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{aligned}\right\} & \text{(conservation of momenta)} \\ (\rho E)_t + (u(\rho E + p))_x + (v(\rho E + p)_y = 0, & \text{(conservation of energy)} \end{cases} \tag{3.1}$$

where ρ, p and (u, v) are the density, pressure and velocity, respectively, $E = \frac{u^2+v^2}{2} + \frac{p}{(\gamma-1)\rho}$ is the total energy with the adiabatic index $\gamma > 1$ for polytropic gases.

$$(\rho, p, u, v)|_{t=0} = (\rho_i, p_i, u_i, v_i) := \text{ⓘ}, \qquad (x, y) \in i\,\text{th quadrant}, \qquad i = 1, 2, 3, 4, \tag{3.2}$$

where $(\rho_i, p_i, u_i, v_i)(1 \le i \le 4)$ are constant states.

We seek self-similar solutions. We shall omit "pseudo-" in pseudo-stationary flow, for simplicity. Then (3.1) and (3.2) change to

$$\begin{cases} -\xi\rho_\xi - \eta\rho_\eta + (\rho u)_\xi + (\rho v)_\xi = 0, \\ -\xi(\rho u)_\xi - \eta(\rho u)_\eta + (\rho u^2 + p)_\xi + (\rho uv)_\eta = 0, \\ -\xi(\rho v)_\xi - \eta(\rho v)_\eta + (\rho uv)_\xi + (\rho v^2 + p)_\eta = 0, \\ -\xi(\rho E)_\xi - \eta(\rho E)_\eta + (u(\rho E + p))_\xi + (v(\rho E + p)_\eta = 0, \end{cases} \tag{3.3}$$

and boundary values at infinity, i.e.,

$$\lim_{\substack{\xi^2 + \eta^2 \to +\infty \\ \eta/\xi = \arctan\theta}} (\rho, p, u, v) = (\rho_i, p_i, u_i, v_i), \quad \frac{\pi}{2}(i-1) < \theta < \frac{\pi}{2}i, \quad 1 \le i \le 4. \tag{3.4}$$

The eigenvalues are

$$\Lambda_0 = \frac{V}{U} \quad \text{(flow characteristic)},$$

$$\Lambda_\pm = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2} \quad \text{(wave characteristics)},$$

where velocity $(U, V) = (u - \xi, v - \eta)$. System 3.3 is of mixed-type, i.e., the flow is transonic:

(a) System (3.3) is hyperbolic (supersonic), if $U^2 + V^2 > c^2$.

(b) System (3.3) is elliptic (subsonic), if $U^2 + V^2 < c^2$.

(c) System (3.3) is parabolically degenerate (sonic), if $U^2 + V^2 = c^2$.

The elementary wave corresponding to Λ_0 is contact discontinuity J which is classified to $J^\pm$ according to $\mathrm{curl}(U, V) = \pm\infty$.

At infinity, there are four jumps, each of them emits three waves: $\overleftarrow{R}$ or $\overleftarrow{S}$, $J^\pm$, $\overrightarrow{R}$ or $\overrightarrow{S}$. The problem is how the twelve waves coming from infinity interact and match together ultimately. Obviously, it is too complicated to deal with but its key point is the interaction of different elementary waves, so they made a restriction that each jump at infinity emits exactly one elementary wave: $\overleftarrow{R}$, $\overrightarrow{R}$, $\overleftarrow{S}$, $\overrightarrow{S}$, J^+ or J^-. According to combinations of the four waves coming from infinity and compatibility, there are 19 final cases [39, 41, 65]. Six of them are fundamental: $\overrightarrow{R}\ \overrightarrow{R}\ \overrightarrow{R}\ \overrightarrow{R}$, $\overrightarrow{R}\ \overleftarrow{R}\ \overleftarrow{R}\ \overrightarrow{R}$, $\overrightarrow{S}\ \overrightarrow{S}\ \overrightarrow{S}\ \overrightarrow{S}$, $\overrightarrow{S}\ \overleftarrow{S}\ \overleftarrow{S}\ \overrightarrow{S}$, $J^-J^-J^-J^-$ and $J^-J^+J^-J^+$.

In order to clarify the boundary of wave interaction domain, let us look at the generalized characteristic analysis of constant state, J^+, $\overrightarrow{R}$ and $\overrightarrow{S}$ through Figure 3.1–3.4.*.

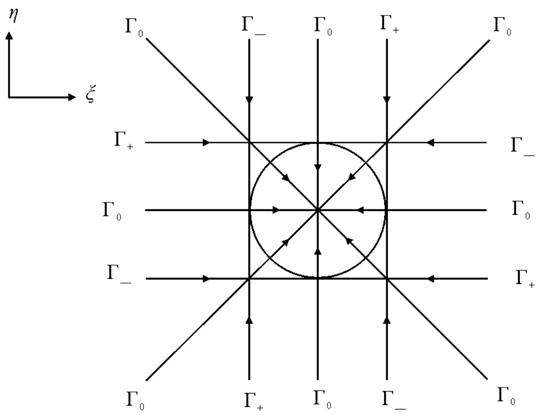


Figure 3.1

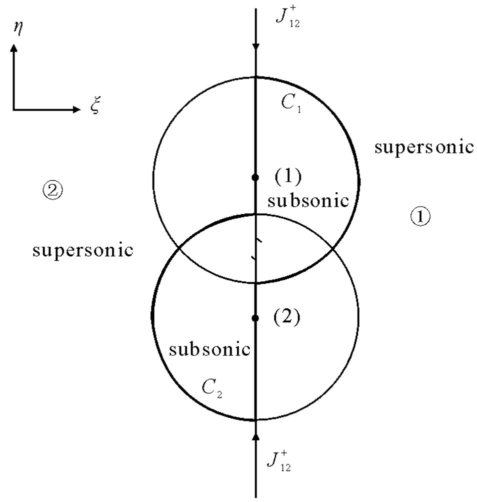


Figure 3.2

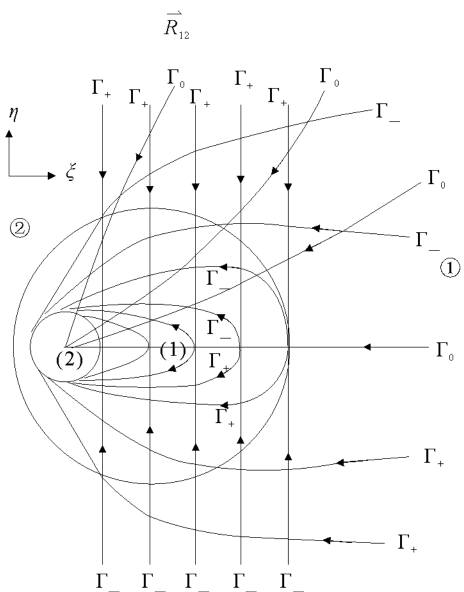


Figure 3.3

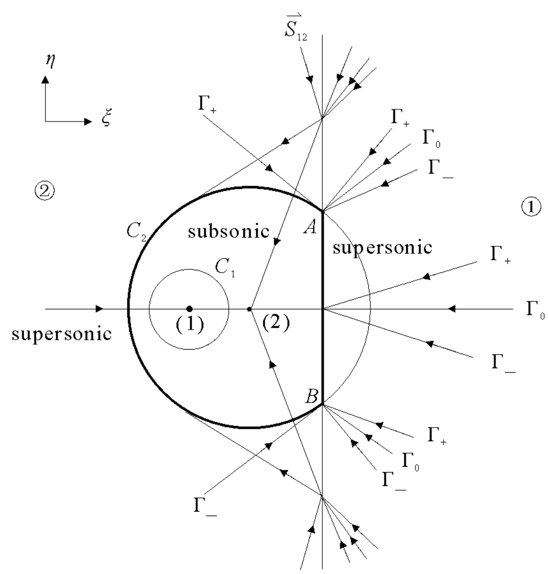


Figure 3.4

Zhang and Zheng constructed the boundaries of interaction of four planar waves coming from infinity case by case. Each boundary consists of characteristics, shocks and sonic curves. Based on the contents of one-dimensional interaction of waves, two-dimensional Riemann problem of scalar conservation laws and reflection of shocks in [13], they formulated a set of conjectures of the wave patterns in the domains of interaction which include expansion of gas, reflection of shocks and formation of spirals (see Figure 3.5).

Schulz-Rinne et. al. [65], Lax and Liu [41], Kurganov and Tadmor [39], Li, Zhang and

*The figures show that the pseudo-stationary flow is very different from the stationary one.

Yang [52] and others performed numerical simulations. Results of the numerical simulations are similar and consistent with the conjectures in over-all appearance except for a missing shock in (a) and a spiral in (b).

Recently, in a joint work with Glimm and other collaborators [33], we found a transonic shock through delicate numerical simulation and generalized characteristic analysis and clarified the mathematical mechanism of shock formation for four $\overrightarrow{R}$ case. This transonic shock has been missing in the conjecture and all of the aforementioned numerical simulations.

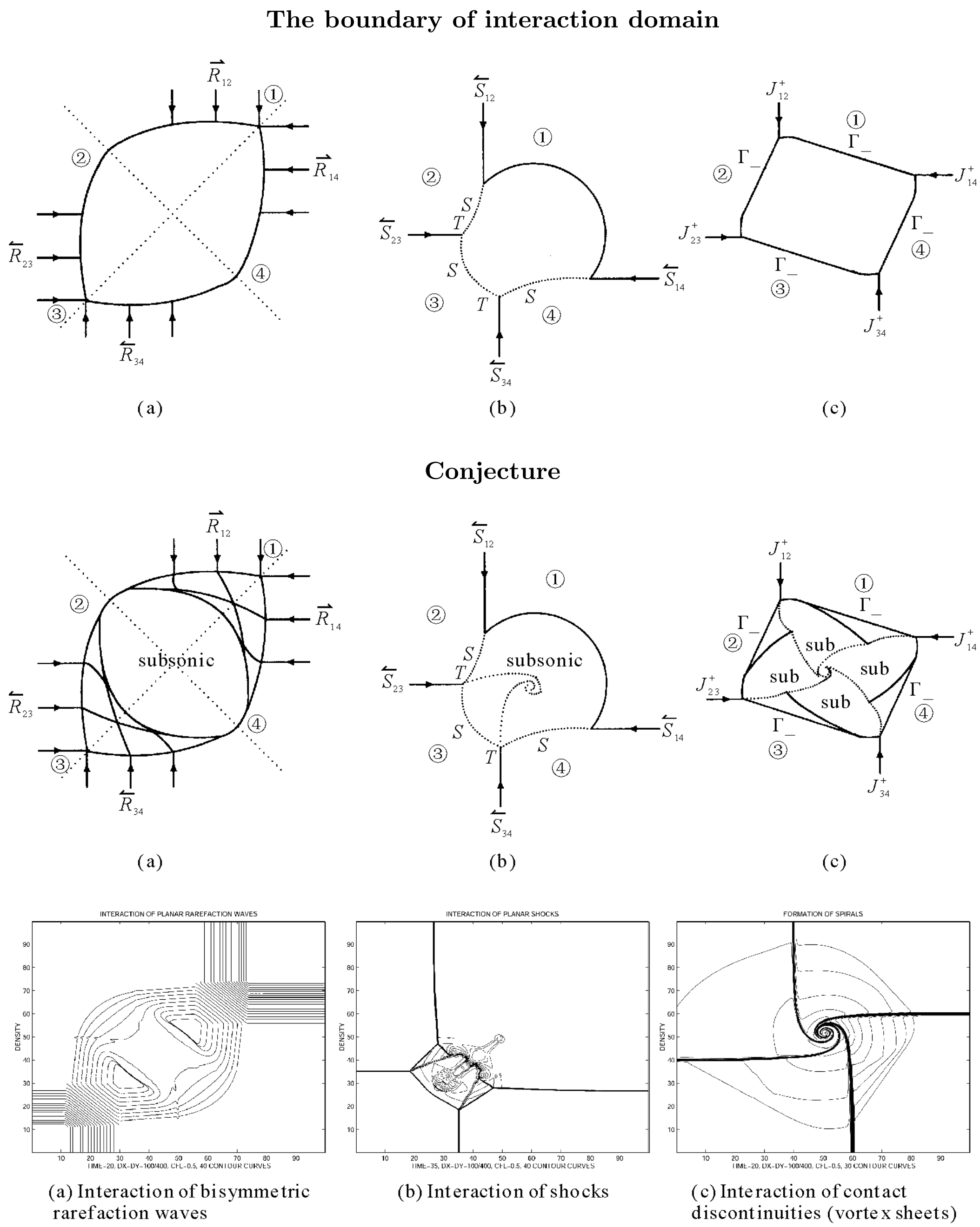


Figure 3.5 Some examples of two-dimensional Riemann solutions

We have been making efforts to rigorously understand wave configurations. The interaction of rarefaction waves is a good starting point. Let us consider the flow near rarefaction waves and the case that no shock is present. Then the flow is isentropic and irrotational. Therefore, (3.1) can be written, in terms of the variables (ξ,η), as

$$\begin{cases} Ui_\xi + Vi_\eta + 2\kappa\, i\,(u_\xi + v_\eta) = 0, \\ Uu_\xi + Vu_\eta + i_\xi = 0, \\ Uv_\xi + Vv_\eta + i_\eta = 0, \end{cases} \tag{3.5}$$

where $(U,V) = (u-\xi, v-\eta)$, $i = \frac{c^2}{\gamma-1}$, and $c^2 = p'(\rho)$. With the ir-rotationality condition

$$u_\eta = v_\xi, \tag{3.6}$$

system (3.5)–(3.6) can further be reduced to

$$\begin{cases} (c^2-U^2)u_\xi - UV(u_\eta + v_\xi) + (c^2 - V^2)v_\eta = 0, \\ u_\eta - v_\xi = 0, \end{cases} \tag{3.7}$$

supplemented by (pseudo-)Bernoulli's law

$$i + \frac{1}{2}(U^2+V^2) = -\varphi, \quad \varphi_\xi = U,\ \varphi_\eta = V. \tag{3.8}$$

From now on, we choose to focus on the discussion in hyperbolic regions. Useful approaches are the methods of characteristic decomposition, hodograph transformation and phase space analysis. In the construction of global flow patterns, main patches are simple waves, "mixed waves" resulted from the binary interaction of simple or planar rarefaction waves, and semi-hyperbolic waves resulted from interaction of simple waves with subsonic regions.

3.1 Characteristic decompositions and simple waves

The idea to use characteristic decompositions can trace back to the classical wave equation $u_{tt} - c^2 u_{xx} = 0$, $c > 0$ is constant. This decomposition is essential not only in the elementary wave patches, but also in providing a passage to derive a apriori estimates of solutions. For the self-similar Euler equations (3.7) we have

Proposition 3.1 (Characteristic decomposition [53]) For (3.7) and (3.8), there hold

$$\partial^+\partial^- I = m\partial^- I, \qquad \partial^-\partial^+ J = n\partial^+ J, \tag{3.9}$$

where m, n can be expressed in the form

$$m = m(u,v)(\partial_\xi u + \zeta_1(u,v)\partial_\eta u),$$

$$n = n(u,v)(\partial_\xi u + \zeta_2(u,v)\partial_\eta u),$$

and $I = u$, v, c or Λ_-, $J = u$, v, c or Λ_+.

With this proposition we show the existence of simple waves adjacent to a constant state, although this is well known for steady flows [23] and one-dimensional hyperbolic conservation laws [25].

Proposition 3.2 (Simple waves [53]) Adjacent to a constant state in the self-similar plane of the adiabatic Euler system is a simple wave in which the physical variables (u, v, c, p, ρ) are constant along a family of wave characteristics which are straight lines, provided that the region is such that its pseudo-flow characteristics extend into the state of constant.

The decomposition indeed has more implications and it can apply to high order estimates of solutions, even to numerical schemes [4].

3.2 Diagonalization

A diagonalization process is always useful in the context of hyperbolic problems. Things are the same here. Introduce the inclination angles α and β of characteristics and the inclination angle τ of streamlines:

$$\tan\alpha := \Lambda_+, \quad \tan\beta := \Lambda_-, \quad \tau = \frac{\alpha+\beta}{2}. \tag{3.10}$$

Then we can diagonalize (3.7) in the form [55],

$$\begin{cases} \bar{\partial}^+(-\beta+\psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega)-\kappa]}{c(\kappa+\sin^2\omega)}, \\ \bar{\partial}^-(\alpha+\psi(\omega)) = \dfrac{\sin^2\omega[\cos(2\omega)-\kappa]}{c(\kappa+\sin^2\omega)}, \\ \bar{\partial}^0[c^2(1+\kappa M^2)] = 2c\kappa M, \end{cases} \tag{3.11}$$

where $\omega = \frac{\alpha-\beta}{2}$ is the Mach angle, $M = \frac{1}{\sin\omega}$ the pseudo-Mach number, $\bar{\partial}^0 = \cos\tau\partial_\xi + \sin\tau\partial_\eta$, $\bar{\partial}^+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta$ and $\bar{\partial}^- = \cos\beta\partial_\xi + \sin\beta\partial_\eta$ are normalized vector fields along characteristics, and

$$\psi(\omega) := \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\cot\omega\right), \tag{3.12}$$

3.3 Hodograph transformation

The approach of hodograph transformation converts the role of dependent variables and independent variables. It is classical for steady flows and it transforms the nonlinear system of steady irrotational flows into a linear form [23]. The pseudo-steady flow (3.7) is much more involved. Pogodin, Suchkov and Ianenko proposed the hodograph transformation [62],

$$T : (\xi, \eta) \to (u, v) \tag{3.13}$$

for (3.5). Then i as the function of u and v satisfies

$$\xi - u = i_u, \qquad \eta - v = i_v, \tag{3.14}$$

provided that the transformation (3.13) is non-degenerate. And i satisfies

$$(2\kappa\, i - i_u^2)i_{vv} + 2i_u i_v i_{uv} + (2\kappa\, i - i_v^2)i_{uu} = i_u^2 + i_v^2 - 4\kappa\, i. \tag{3.15}$$

The linear degeneracy of 1.29 becomes more transparent when it is expressed in terms of α, β and c. In paper [54], we convert (3.15) to

$$\begin{cases} \bar{\partial}_+\alpha = \dfrac{1+\gamma}{4c}\cdot\sin(\alpha-\beta)\cdot\left[m-\tan^2\omega\right] =: G(\alpha,\beta,c), \\ \bar{\partial}_-\beta = G(\alpha,\beta,c), \\ \partial_0 c = \kappa\cos\dfrac{\alpha+\beta}{2}/\sin\omega, \end{cases} \tag{3.16}$$

with $\bar{\partial}_+ = (\sin\beta, -\cos\beta)\cdot(\partial_u, \partial_v)$, $\bar{\partial}_- = (\sin\alpha, -\cos\alpha)\cdot(\partial_u, \partial_v)$, $\partial_0 = \partial_u$ (noting the differences from $\bar{\partial}^{\pm}$, $\bar{\partial}^0$ in the (ξ,η)-plane), and

$$\bar{\partial}_+ c = -\kappa, \quad \bar{\partial}_- c = \kappa. \tag{3.17}$$

The definitions of $\bar{\partial}_{\pm}$ and ∂_0 implies that system (3.16) is linearly degenerate.

In our construction of solutions, we need C^0, C^1 and $C^{1,1}$ estimates. The main difficulty lies in the non-homogeneity of (3.16). Thus we shall need the second-order derivatives, given in [54]. These formula are somewhat like the characteristic decompositions (3.9) in the (ξ,η)-plane.

Proposition 3.3 Assume that the solution of (3.16) $(\alpha,\beta) \in C^2$. Then

(i) For the inclination angles α and β, we have

$$\begin{cases} \bar{\partial}_+\bar{\partial}_-\alpha + W\bar{\partial}_-\alpha = Q(\omega,c), \\ -\bar{\partial}_-\bar{\partial}_+\beta + W\bar{\partial}_+\beta = Q(\omega,c), \end{cases} \tag{3.18}$$

where $W(\omega,c)$ and $Q(\omega,c)$ are

$$\begin{aligned} W(\omega,c) &:= \frac{1+\gamma}{4c}\Big[\left(m-\tan^2\omega\right)\left(3\tan^2\omega-1\right)\cos^2\omega + 2\tan^2\omega\Big], \\ Q(\omega,c) &:= \frac{(1+\gamma)^2}{16c^2}\sin(2\omega)\left(m-\tan^2\omega\right)\left(3\tan^2\omega-1\right). \end{aligned} \tag{3.19}$$

(ii) For the inclination angle τ of streamlines we have

$$\begin{cases} \bar{\partial}_+\bar{\partial}_-\tau + W\bar{\partial}_-\tau = a(\omega,c)\bar{\partial}_+\tau \\ -\bar{\partial}_-\bar{\partial}_+\tau + W\bar{\partial}_+\tau = a(\omega,c)\bar{\partial}_-\tau, \end{cases} \tag{3.20}$$

where

$$a(\omega,c) := \frac{\gamma+1}{4c}\cos^2\omega(\tan^2\omega+\alpha_2)(\tan^2\omega-\alpha_1), \tag{3.21}$$

and

$$\alpha_2 := \frac{1}{2}[3+m+\sqrt{(3+m)^2+4m}], \quad \alpha_1 := \frac{2m}{3+m+\sqrt{(3+m)^2+4m}}. \tag{3.22}$$

To invert the solution on the hodograph plane to the (ξ,η) plane, we notice that (3.14) defines a mapping from $(u,v,)$ to (ξ,η) as $\xi = u + i_u, \eta = v + i_v$. The Jacobian has the formula

$$j(\xi,\eta;u,v) = \frac{\partial(\xi,\eta)}{\partial(u,v)} = \frac{c^2}{4\sin^4\omega}(\bar{\partial}_-\alpha - Z)(\bar{\partial}_+\beta - Z) \tag{3.23}$$

where

$$Z := \frac{\gamma+1}{2c}\tan\omega. \tag{3.24}$$

Proposition 3.4 Assume that the solution of (3.16) $(\alpha,\beta) \in C^2$. Then we have

$$\begin{cases} (\bar{\partial}_+ + W)(Z - \bar{\partial}_-\alpha) = \dfrac{\gamma+1}{4c}(\tan^2\omega+1)(Z-\bar{\partial}_+\beta) \\ (-\bar{\partial}_- + W)(Z - \bar{\partial}_+\beta) = \dfrac{\gamma+1}{4c}(\tan^2\omega+1)(Z-\bar{\partial}_-\alpha). \end{cases} \tag{3.25}$$

3.4 A direct method

In order to avoid the complexity of the hodograph transform near sonic curves, shocks or some physical boundaries, a direct method is developed [20, 50]. This method is based on the following decompositions, in addition to the diagonal system (3.11).

Proposition 3.5 (Characteristic decomposition for c in the (ξ,η)-plane) For the variable c, we have the following characteristic decompositions,

$$\begin{cases} c\bar{\partial}^+\bar{\partial}^- c = \bar{\partial}^- c\Big\{-\sin(2\omega) + \dfrac{1}{2\mu^2\cos^2\omega}\bar{\partial}^- c + \Big[\dfrac{\Omega\cos(2\omega)}{2\mu^2}+1\Big]\bar{\partial}^+ c\Big\}, \\ c\bar{\partial}^-\bar{\partial}^+ c = \bar{\partial}^+ c\Big\{-\sin(2\omega) + \dfrac{1}{2\mu^2\cos^2\omega}\bar{\partial}^+ c + \Big[\dfrac{\Omega\cos(2\omega)}{2\mu^2}+1\Big]\bar{\partial}^- c\Big\}, \end{cases} \tag{3.26}$$

where $\Omega = \frac{3-\gamma}{\gamma+1} - \tan^2\omega$ and $\mu^2 = \frac{\gamma-1}{\gamma+1}$.

We notice that c is related to other state variables via the following identities.

$$\begin{aligned} &\bar{\partial}^- u = \frac{\sin\alpha}{\kappa}\bar{\partial}^- c, \quad \bar{\partial}^+ u = -\frac{\sin\beta}{\kappa}\bar{\partial}^+ c, \\ &\bar{\partial}^- v = -\frac{\cos\alpha}{\kappa}\bar{\partial}^- c, \quad \bar{\partial}^+ v = \frac{\cos\beta}{\kappa}\bar{\partial}^+ c. \end{aligned} \tag{3.27}$$

$$\begin{cases} c\bar{\partial}^+\alpha = -\dfrac{\sin(2\delta)}{2\mu^2}\Omega\bar{\partial}^+ c, \\ c\bar{\partial}^+\beta = -\dfrac{\tan\delta}{\mu^2}\bar{\partial}^+ c + 2\sin^2\delta, \end{cases} \qquad \begin{cases} c\bar{\partial}^-\alpha = \dfrac{\tan\delta}{\mu^2}\bar{\partial}^- c - 2\sin^2\delta, \\ c\bar{\partial}^-\beta = \dfrac{\sin(2\delta)}{2\mu^2}\Omega\bar{\partial}^- c. \end{cases} \tag{3.28}$$

Then we can obtain characteristic decompositions for all state variables.

The direct method has potential to extend the solution to sonic curves (degenerate boundaries) or shocks.

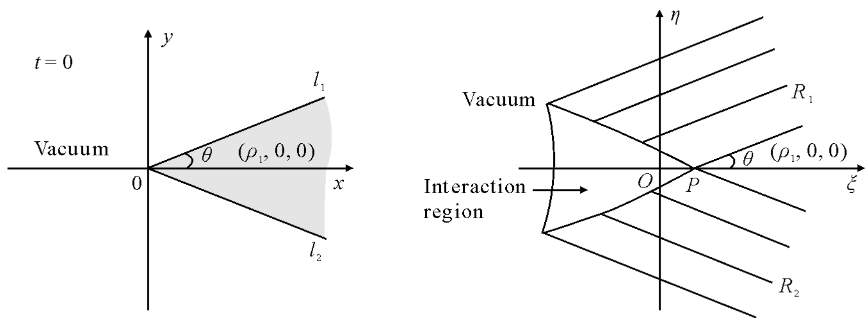
3.5 Binary interaction of planar rarefaction waves

The interaction of two planar rarefaction waves has been proven to be fundamental in the construction of general continuous solutions. In fact, such an interaction is in its own important. It contains the dam collapse problem ($\gamma = 2$ for shallow waters) as a specific example. Earlier analytic studies about this problem were traced back to [62] in the 1950's and later to [43, 73] in the 1960's. In [45, 46] this problem was investigated in the hodograph plane for $1 \le \gamma \le 3$, and then in [54] the solution was fully transformed back to the physical self-similar plane. This completes a circle to study the interaction problem of planar rarefaction waves. This problem has also been solved by using the direct method [20, 52] recently.

We place the wedge symmetrically with respect to the x-axis and the sharp corner at the origin, as in Figure 3.6 (a). This problem is then formulated mathematically as seeking the solution with the initial data,

$$(i,u,v)(t=0,x,y) = \begin{cases} (i_0,u_0,v_0), & -\theta < \delta < \theta, \\ (0,\bar{u},\bar{v}), & \text{otherwise}, \end{cases} \tag{3.29}$$

where $i_0 > 0$, u_0 and v_0 are constant, $(\bar{u},\bar{v})$ is the velocity of the wave front, not being specified in the state of vacuum, $\delta = \arctan(y/x)$ is the polar angle, and θ is the half-angle of the wedge restricted between 0 and $\pi/2$. This can be considered as a two-dimensional Riemann problem with two pieces of initial data (3.29) and falls into the framework of the interaction of two whole planar rarefaction waves. See Figure 3.6 (b). We note that the solution we construct is valid for any "portions" of (3.29) as the solutions are hyperbolic.



(a) Initial Data (b) Interaction of rarefaction waves

Figure 3.6 The expansion of a wedge of gas

Theorem 3.6 (Gas expansion [54]) There exists a solution $(u, v, \rho) \in C^1$ for the problem of gas expansion into a vacuum in the wave interaction region in the self-similar (ξ, η)-plane for all $\gamma \geq 1$ and all wedge half-angle $\theta \in (0, \pi/2]$. For $\theta > \theta_s := \arctan(\mathrm{Re}(\sqrt{\frac{3-\gamma}{\gamma+1}}))$, the vacuum boundary is representable as a single-valued concave function $\xi = B(\eta)$, the minus family of characteristics are concave, the plus family of characteristics are convex, and the difference of their inclination angles at the boundary is $2\theta_s(\gamma)$.

3.6 Interaction of bi-symmetric planar rarefaction waves

Once we solve the binary interaction of rarefaction waves, we go on our efforts to march towards our goal: To provide a global analytic solution to one of the four-wave Riemann problems. This complete solution is a subcase of the wave structure $\overrightarrow{R}\ \overleftarrow{R}\ \overleftarrow{R}\ \overrightarrow{R}$, or in Figure 3.5 (a). We call it a bi-symmetric interaction of planar rarefaction waves. The initial data satisfies

$$\rho_1 = \rho_3, \quad \rho_2 = \rho_4, \quad u_1 - u_2 = v_1 - v_4, \quad (\rho_2 < \rho_1). \tag{3.30}$$

And the set-up is symmetric with respect to

$$\xi - \eta = u_1 - v_1, \quad \text{and} \quad \xi + \eta = u_2 + v_2.$$

We require the data to satisfy the compatibility condition so that only a single planar rarefaction wave emits from each initial discontinuity [52, 84]. The solution structure is shown in Figure 3.7 and the main result is stated as follows:

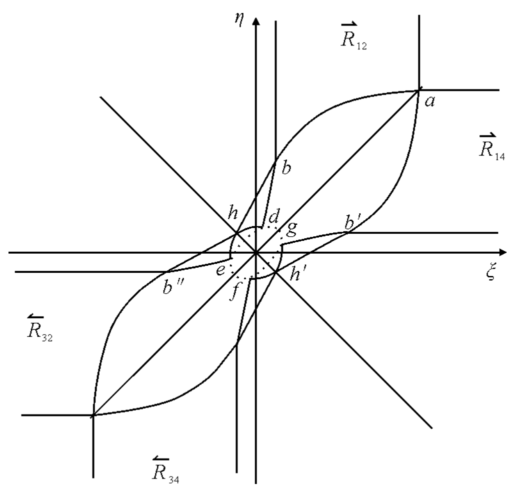


Figure 3.7 Interaction of four bi-symmetric rarefaction waves

Theorem 3.7 (Bi-symmetric rarefactive Riemann solution [55]) Consider the Riemann problem for system (3.1) with initial data of the type $\overrightarrow{R}\ \overleftarrow{R}\ \overleftarrow{R}\ \overrightarrow{R}$. Then, there exists a number $c_2^*(\gamma) \in (0,1)$ for $\gamma > 1+\sqrt{2}$, such that our bi-symmetric Riemann problem has global continuous solutions, provided $0 < c_2 < c_2^*(\gamma)c_1$.

This solution is shown in Figure 3.7, where vacuum appears in the region "defg". The solution can be regarded as the first analytic global solution for all four-wave Riemann problems.

3.7 Semi-hyperbolic patches

If c_2 violates the smallness condition in Theorem 3.7, then the solution to the bi-symmetric four rarefaction wave interaction is likely to have a nontrivial subsonic domain at the center, so that the solution is transonic, which is shown in Figure 3.5 (a). We observe that the transition from supersonic to subsonic occurs on both a shock curve and a continuous curve in Figure 3.5 (a). It is a substantial challenge to handle such a case. In papers [56, 72], the authors experiment construction of solutions just outside the subsonic domain. The solution there is not in the domain of determinacy of the binary interactions R_{12} with R_{14} or R_{23} with R_{34} in the classical sense. The true solution really depends on the solution of the subsonic domain and the free sonic boundary or shock boundary, but the authors of [56, 72] propose a family of artificial Goursat problems whose solutions are expected to capture the true solution. The artificial Goursat problems and their solutions are presented in the following theorem for the Euler equations from paper [56].

Theorem 3.8 (Semi-hyperbolic patch [56]) Consider a positive characteristic curve $\widehat{BA}$ in a simple wave of straight negative characteristic curves, see Figure 3.8, and a strictly convex negative characteristic extension $\widehat{BC}$, forming a Goursat problem for a possible solution in the domain ABC. Then there exists a smooth solution in ABC, called a semi-hyperbolic patch, as shown in Figure 3.8. The positive and negative characteristics are either concave and convex respectively. The boundary $\widehat{AC}$ is Lipschitz and sonic. Furthermore, the positive characteristics starting on the curve $\widehat{BC}$, downward, form an envelope before reaching their sonic points.

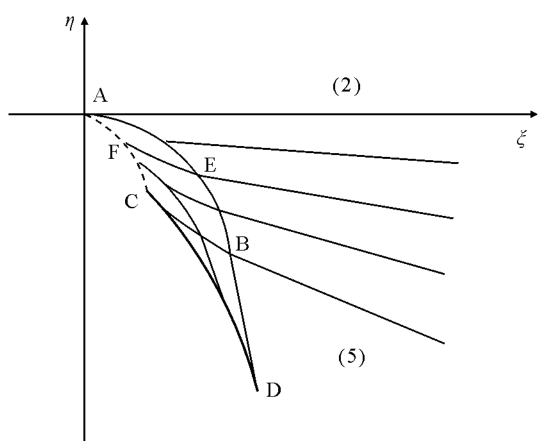


Figure 3.8 Set-up for a semi-hyperbolic patch in a simple wave. Given a positive characteristic $\widehat{BEA}$ in a simple wave and a strictly convex negative characteristic $\widehat{BC}$, find a solution in curvilinear triangle ABC, where AFC is sonic while CD is an envelope. Here state (5) represents the state after the interaction of R_{12} with R_{14}.

The position $\widehat{BC}$ is arbitrary, except it is convex, for the existence of such a patch, and we believe that one of such $\widehat{BC}$'s would give the true solution reflected in the numerical simulation of Figure 3.5 (a).

3.8 Axisymmetric solutions of Euler equations

The Euler equations subject to the initial data

$$(\rho, u, v)(t=0, x, y) = (\rho_0, \quad u_0 \cos\theta + v_0 \sin\theta, \quad u_0 \sin\theta - v_0 \cos\theta) \tag{7.1}$$

was investigated in [85, 86], where ρ_0, u_0 and v_0 are constants and θ is the polar angle. The problem is transferred into solving the problem

$$\begin{aligned} &(\rho_r, u_r, v_r) = \left(\frac{\rho}{r} \frac{v^2 - u(r-u)}{p'(\rho) - (u-r)^2}, \quad \frac{1}{r} \frac{v^2(r-u) - up'(\rho)}{p'(\rho) - (u-r)^2}, \quad \frac{uv}{r(r-u)} \right), \\ &\lim_{r \to +\infty} (\rho, u, v) = (\rho_0, \quad u_0 \cos\theta + v_0 \sin\theta, \quad u_0 \sin\theta - v_0 \cos\theta). \end{aligned} \tag{7.2}$$

This is a problem to seek a trajectory connecting the infinity to the origin. The results disclose many interesting patterns, such as spirals, shock waves, expansion waves and the vacuum. The detailed proof can be found in [88].

It is interesting to note that an analytical spiral solution is obtained

$$u = \frac{1}{2} r, \ v = \frac{1}{2} r, \ \sqrt{p'(\rho)} = \frac{1}{2} r, \ \text{for } 0 < r < \frac{1}{2} \Big(\sqrt{p'(\rho_0) + 4v_0^2} + \sqrt{p'(\rho_0)} \Big), \tag{3.31}$$

for $\gamma = 2$ and initial data $u_0 = 0$ and $M_0 = v_0 / \sqrt{p'(\rho_0)} = \sqrt{2}$. This explicit solution may be useful as a testing case in numerical simulations of tornadoes.

3.9 Reflection of oblique shocks

The reflection of oblique shocks can be regarded as a special family of two-dimensional Riemann problems. The initial data is set as

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (\rho_1, u_1, 0, p_1), & \text{for } x < 0, y > 0, \\ (\rho_0, u_0, v_0, p_0), & \text{for } x > 0, y/x > \tan\theta, \end{cases} \tag{3.32}$$

and $(\rho_1, u_1, 0, p_1)$ and (ρ_0, u_0, v_0, p_0) are so chosen that $v_0/u_0 = \tan\theta$ and a single shock initially emitting from $x = 0$ moves to the right and hits the rigid wall $y/x = \tan\theta$. The boundary condition is

$$v(x, y, t) = \begin{cases} 0, & \text{on } x < 0, y = 0, \\ u \tan\theta, & \text{on } x > 0, y = x \tan\theta. \end{cases} \tag{3.33}$$

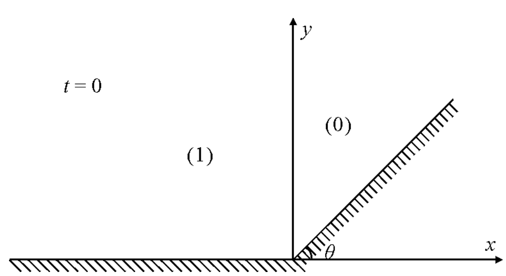
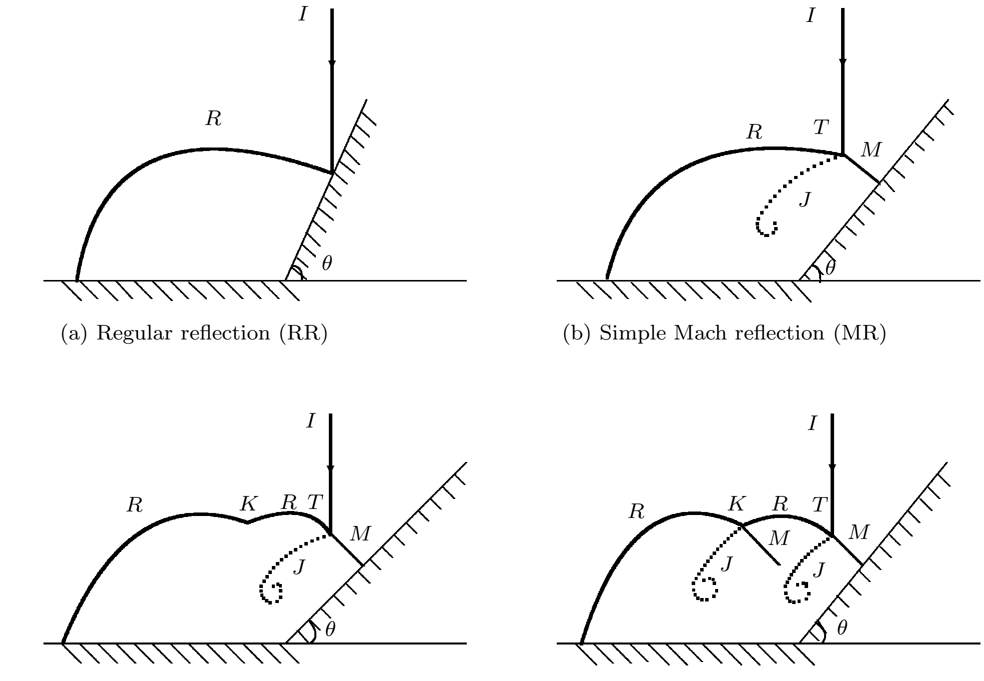


Figure 3.9 Initial data for the oblique shock reflection

There are many patterns of solutions to (3.1), (3.32) and (3.33), known as the regular reflection, simple Mach reflection (MR), complex Mach reflection (CMR), double Mach reflection (DMR) and probably some other types of reflections. See Figure 3.10 or [5]. Justification of each of these patterns comes mainly from numerical simulations and physical experiments. We refer the readers to Ben-dor and Glass [5] for further details.



(a) Regular reflection (RR)

(b) Simple Mach reflection (MR)

(c) Complex Mach reflection (CMR)

(d) Double Mach reflection (CMR)

Figure 3.10 Patterns of solutions to the oblique shock reflection problem

Analytic justification is extremely difficult and it is still at the stage of algebraic analysis via the shock polar, see [23]. Bleakney and Taub [6] delivered an analytical formulation of the criterion between regular reflection and Mach reflection. The formulation is very complicated and completed by Chang and Chen [12] and Sheng and Yin [70] at last. Using the full Euler equations (3.1), Chen [19] verified the stability of a special class of simple Mach reflections. Chen and Feldman [15], Elling [28] and Elling and Liu [29] established the existence of solutions to potential flow equations for the regular reflection problem of oblique shock. Zheng [93] showed the existence of solutions to the case of regular reflections for the full Euler system as the inclination angle of the rigid wall is close to $\pi/2$.

4 Simplified Models

4.1 Pressure-gradient equations

The pressure-gradient equations can be obtained, together with the zero-pressure equations in the next section, by the operator splitting method for the Euler equations (3.1), see [1, 52, 57, 88]. Another derivation is from some asymptotic analysis [91]. The final form after re-scaling is

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (pu)_x + (pv)_y = 0, \end{cases} \tag{4.1}$$

where the notations have the same meanings as those of Euler equations, except $E = (u^2 + v^2)/2 + p$. This system can approximate the Euler system when the Mach number or pressure gradient is large, and it is very close to the classical wave equation. In fact, we have

$$\left(\frac{p_t}{p}\right)_t - p_{xx} - p_{yy} = 0. \tag{4.2}$$

In terms of self-similar variables, this equation can be written as

$$(p-\xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p-\eta^2)p_{\eta\eta} + \frac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0.$$

We are more interested in this equation because it is a purely scalar equation of p, and the slight nonlinearity allows more rigorous analysis.

In [81], the authors mimicked [84] to analyze the two-dimensional Riemann problem for (4.1). The solution structures are strikingly analogous to those of Euler equations except those involving contact discontinuities. Then the analysis was followed by numerical simulations.

Equation (4.1) or the self form of (4.1) is of mixed-type, which we can see from their eigenvalues

$$\Lambda_\pm = \frac{\xi\eta \pm \sqrt{p(\xi^2+\eta^2-p)}}{\xi^2 - p}. \tag{4.3}$$

Zheng in [87] proved the following theorem concerning the existence of solutions in subsonic-sonic regions. Let Ω be any open, bounded, and convex domain with a $C^{2,\alpha}$ boundary ($\alpha \in (0,1)$) that does not contain the origin.

Theorem 4.1 (Existence of subsonic solutions [87]) Consider (4.1) inside the domain Ω with the boundary data $p|_{\partial\Omega} = \xi^2 + \eta^2$. Then there exists a positive weak solution $p \in H^1_{\mathrm{loc}}(\Omega)$ with $p \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$. It takes on the boundary value in the sense $[p - (\xi^2+\eta^2)]^{3/2} \in H^1_0(\Omega)$. Furthermore, it has

(i) maximum principle: $\min_{\partial\Omega}(\xi^2+\eta^2) \le p(\xi,\eta) \le \max_{\partial\Omega}(\xi^2+\eta^2)$;

(ii) interior ellipticity: $p(\xi,\eta) - (\xi^2+\eta^2) > 0$ in Ω.

Song in [71] has removed the restriction on the origin and the complete smoothness of the boundary; and Kim and Song in [36] have obtained regularity of the solution in the interior of the domain and continuity up to and includes the boundary.

In hyperbolic regions $\xi^2 + \eta^2 > p$, this equation also has characteristic decompositions, similar to those for (3.7). In fact, we let

$$r = \sqrt{\xi^2+\eta^2}, \quad \theta = \arctan(\eta/\xi), \quad \lambda = \sqrt{\frac{p}{r^2(r^2-p)}},$$

and

$$\partial_\pm = \partial_\theta + \frac{1}{\lambda}\partial_r.$$

Then we have

$$\begin{cases} \partial_+\partial_- = mp_r\partial_- p, \\ \partial_-\partial_+ = -mp_r\partial_+ p, \end{cases} \tag{4.4}$$

where $m = \frac{\lambda r^4}{2p^2}$. We can also use the coordinates (ξ, η) to derive similar characteristic decompositions.

Using these decompositions, we can discuss the solution to (4.1) in hyperbolic regions. Dai and Zhang solved the problem of planar rarefaction wave interaction in [26] with a vacuum bubble near the origin, and then Lei and Zheng in [42] showed that the vacuum bubble is imaginary that does not exist in reality. Yang and Zhang solved this problem in the hodograph plane [79]. The result is stated as follows.

Theorem 4.2 (Existence [26, 42]) There is a classical smooth solution to the interaction of planar rarefaction waves for the pressure gradient equations (4.1). See Figure 4.1. This also applies to the case $0 < \theta \leq \pi/2$.

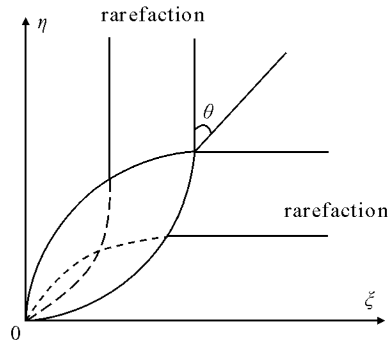


Figure 4.1 Interaction of binary planar rarefaction waves for pressure gradient equations

Based on this, Bang in [2] considered the interaction of three and four rarefaction waves, and established the existence of solution and provided explicit solution structures.

For the Riemann problem involving shocks, we have to solve transonic problem with shocks as free boundaries. Along this direction Zheng in [89] provided a global solution to the case that has initially two contact discontinuities plus two shocks. This is just Configuration H in [52, Page 206]. Then Zheng in [90] proved the existence of solutions to the regular reflection problem of shocks. Almost immediately, regular reflection problems for the isentropic and irrotational flow were established [15, 28, 29].

4.2 Zero-pressure gas dynamics and Delta-shocks

In this section we will discuss a class of hyperbolic conservation laws: The zero-pressure gas dynamics. This system can be derived from a splitting algorithm for the Euler equations (3.1), see [1, 57]; or via the procedure letting the temperature drop to zero [44]. Later approximation results can be found in [17, 18]. Indeed, this model has its own physical meaning and explains the formation of large scale universe in astrophysics [38, 66]. Therefore we call it a sticky particle model. It is also interesting to see its relation with the Boltzmann equation [8, 52] and the Hamilton-Jacobi equation [9].

We will not summarize the interesting well-posedness of the Cauchy problem for this model, although it was well established [21, 27, 30, 48] or its statistical properties [27]. Instead, we just concentrate on the solution structures via the Riemann problem.

The zero-pressure gas dynamics model has the form

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \end{cases} \tag{4.5}$$

where ρ is the density and $U = (u, v)$ the velocity. The distinct feature from classical hyperbolic conservation laws is that the density and momentum may become a singular measure [51, 52]. Two blow-up mechanisms occur simultaneously. So we have to consider the solutions in a much broader sense.

Definition 4.3 ([8, 52]) Denote by $BM(\mathbb{R}^n)$ $(n = 1, 2)$ the space of bounded Borel measure on $\mathbb{R}^n$. Consider a mass distribution $\rho(t, \cdot)$ and a momentum distribution $Q(t, \cdot)$ satisfying

$$\begin{aligned} \rho(t, \cdot) \in C([0, T], BM(\mathbb{R}^n)), \quad \rho \geq 0, \\ Q(t, \cdot) \in C([0, T], (BM(\mathbb{R}^n))^n), \end{aligned} \tag{4.6}$$

such that

$$|Q(t, \cdot)| \leq C\rho(t, \cdot), \quad \forall t \in [0, T], \tag{4.7}$$

in the sense of measures, for a finite constant C. By Radon-Nykodym theorem, there is a mean velocity $U(t, \cdot) \in L^\infty(\rho(t, \cdot))$ for each fixed $t \geq 0$ such that

$$U(t, x) = \frac{\mathrm{d}Q}{\mathrm{d}\rho}. \tag{4.8}$$

Thus, we say a pair $(\rho, U)(t, \cdot)$ is a measure solution of (5.1) if and only if

$$\begin{cases} \displaystyle\int_0^T \int_{\mathbb{R}^n} (\phi_t + U \cdot \nabla\phi) \mathrm{d}\rho \mathrm{d}t = 0, \\ \displaystyle\int_0^T \int_{\mathbb{R}^n} U(\psi_t + U \cdot \nabla\psi) \mathrm{d}\rho \mathrm{d}t = 0, \end{cases} \tag{4.9}$$

for all $\phi, \psi \in C_C^\infty([0, T] \times \mathbb{R}^n)$.

Remark 4.4 Another definition of measure solutions can be made via the Lebesgue-Stieljes integral [30, 77]. It is equivalent to the above definition.

Based on this definition, we define a piecewise smooth solution with a delta-measure supported on a time-space surface,

$$(\rho, U)(t, X) = \begin{cases} (\rho_1, U_1)(t, X), & (t, X) \in \Omega_1, \\ (w(t, s)\delta(X - X(t, s)), U_\delta(t, s)), & (t, X) \in S, \\ (\rho_2, U_2)(t, X), & (t, X) \in \Omega_2, \end{cases} \tag{4.10}$$

where $X = (x, y)$, (ρ_i, U_i) are smooth in Ω_i, $i = 1, 2$, respectively, $s > 0$ is a parameter and δ is a delta measure supported on a smooth surface $X = X(t, s)$, the weight $w(t, s)$ can be regarded as the mass of "big" particles assembling together on the surface. Such a solution is called a delta-shock. The delta-shock solution was first defined in an unpublished Ph. D thesis [37], and it was independently defined for a Burgers-type system in [74].

The delta-shock solution (4.10), just like the classical shocks, is described algebraically via a (generalized) Rankine–Hugoniot jump condition [51],

$$\begin{cases} \dfrac{\partial X}{\partial t} = U_\delta(t,s), \\ \dfrac{\partial w(t,s)}{\partial t} = ([\rho],[\rho U]) \cdot (n_t, n_X), \\ \dfrac{\partial (wU_\delta)(t,s)}{\partial t} = ([\rho U],[\rho U \otimes U]) \cdot (n_t, n_X), \end{cases} \tag{4.11}$$

where $[q] = q_1 - q_2$ denotes the jump of q across the discontinuity, and $n = (n_t, n_X)$ is oriented from Ω_1 to Ω_2. This condition represents the dynamics of the support $X = X(t,s)$, the mass $w(t,s)$ and the momentum $wU_\delta(t,s)$. As $w(t,s) \equiv 0$, this condition is consistent with the classical Rankine–Hugoniot condition for classical shocks.

Also we impose the following entropy condition to guarantee the stability,

$$U_2 \cdot n_X < U_\delta \cdot n_X < U_1 \cdot n_X, \tag{4.12}$$

where the projection n_X of n is oriented from Ω_1 side to Ω_2 side. The stability can further be justified by taking the viscosity vanishing process,

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) = \epsilon t \Delta U, \end{cases} \tag{4.13}$$

where $\epsilon > 0$ is small viscosity, Δ is Laplacian. This was done in the self-similar plane [68] and (t,X)-space [49] with a self-similar viscosity, respectively. This kind of viscosity vanishing method mimicks the traditional manner for shocks. An earlier (more original) result for the system of Burgers-type equations can be found in [75].

The Riemann solution of (4.5) basically contains two families of solutions: the vacuum solution and the delta-shock solution (see Fig.4.2. for 2-D case in ([68])). The one-dimensional Riemann solution can be founded in [8, 47, 68]. Two-dimensional Riemann problems for (4.5) was initiated in [68] and more solution structures were presented in [52, Chapter 3].

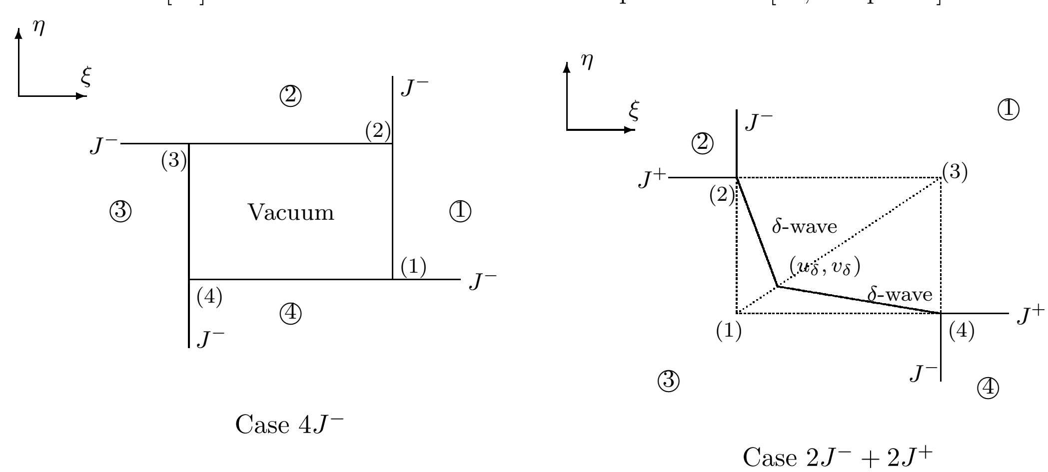


Figure 4.2

Acknowledgements The authors would like to thank Mr. Lihui Guo for ardent help.

References

[1] Agarwal R K, Halt D W. A modified CUSP scheme in wave/particle split form for unstructured grid Euler flow//Caughey D A, Hafez M M, Eds. Frontiers of Computational Fluid Dynamics. New York: Wiley, 1994: 155–163

[2] Bang Seunghoon. Interaction of three and four rarefaction waves of the pressuregradient system. Journal of Differential Equations, in press, 2008

[3] Ben-Artzi M, Falcovitz J, Li J -Q. Wave interactions and numerical approximation for two-dimensional scalar conservation laws. Computational Fluid Dynamics Journal, 2006, **14**: 401–418

[4] Ben-Artzi M, Li J -Q, Warnecke G. A direct Eulerian GRP scheme for compressible fluid flows. J Comput Phys, 2006, **218**: 19–43

[5] Ben-dor G, Glass I I. Domains and boundaries of non-stationary oblique shock wave reflection (1). J Fluid Dynamics, 1979, **92**: 459–496; (2). J Fluid Dynamics, 1980, **96**: 735–756

[6] Bleakney W, Taub A H. Interaction of shock waves. Review of Moder Physics, 1949, **21**: 584–605

[7] Bianchini S, Bressan A. Vanishing viscosity solutions of nonlinear hyperbolic systems. Ann Math, 2005, **161**(1): 223–342

[8] Bouchut F. On zero pressure gas dynamics//Perthame B, ed. Advances in Kinetic Theory and Computing, Series on Advances in Mathematics for Applied Sciences, Vol 22. Singapore: World Scientific, 1994: 171–190

[9] Brenier Y, Grenier E. Sticky particles and scalar conservation laws. SIAM J Numer Anal, 1998, **35**: 2317–2328

[10] Canic S, Keyfitz B L, Kim Eun Heui. A free boundary problem for a quasi-linear degenerate elliptic equation: regular reflection of weak shocks. Comm Pure Appl Math, 2002, **55**: 71–92

[11] Canic S, Keyfitz B. Quasi-one-dimensional Riemann problems and their role in self-similar two-dimensional problems. Arch Rational Mech Anal, 1998, **144**: 233–258

[12] Chang T, Chen G Q. Diffration of planar shock along a compressive corner. Acta Mathematia Scientia, 1986, **6**: 241–257

[13] Chang T, Chen G -Q, Yang S -L. On the 2-D Riemann problem for the compressible Euler equations, I. Interaction of shocks and rarefaction waves. Discrete continuous dynamical systems, 1995, **1**: 555–584; II. Interaction of contact discontinuities. Discrete Contin Dynam Systems, 2000, **6**: 419–430

[14] Chang T, Hsiao L. The Riemann problem and interaction of waves in gas dynamics//Pitman Monographs and Surveys in Pure and Applied Mathematics 41. Harlow: Longman Scientific and Technical, 1989

[15] Chen G -Q, Feldman M. Global solutions of shock reflection by large-angle wedges for potential flow. Ann Math, to appear, 2007

[16] Chen G -Q, Li D, Tan D -C. Structure of Riemann solutions for 2-Dimensional scalar conservation law. J Diff Equat, 1996, **127**: 124–147

[17] Chen G -Q, Liu H -L. Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids. SIAM J Math Anal, 2003, **34**: 925–93

[18] Chen G -Q, Liu H -L. Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids. Phys D, 2004, **189**: 141–165

[19] Chen S -X. Mach configuration in pseudo-stationary compressible flow. J Amer Math Soc, 2008, **21**: 63–100

[20] Chen X, Zheng Y -X. The interaction of rarefaction waves of the two-dimensional Euler equations. Indiana Univ Math J, (in press)

[21] Cheng S, Li J -Q, Zhang T. Explicit construction of measure solutions of the Cauchy problem for the transportation equations. Science in China (Series A), 1997, **40**: 1287–1299

[22] Colella P, Henderson L F. The von Neumann paradox for the diffration of weak shock waves. J Fluid Mech, 1990, **213**: 71–94

[23] Courant R, Friedrichs K O. Supersonic Flow and Shock Waves. New York: Interscience, 1948

[24] Ding X X, Chen G -Q, Luo P -Z. Convergence of the Lax-Friedrichs scheme for the system of equations of isentropic gas dynamics I. Acta Math Sci (Chinese), 1987, **7**: 467–480; II. 1988, **8**: 61–94

[25] Dafermos C. Hyperbolic Conservation Laws in Continuum Physics (Grundlehren der Mathematischen Wissenschaften). Berlin, Heidelberg, New York: Springer, 2000

[26] Dai Z, Zhang T. Existence of a global smooth solution for a degenerate Goursat problem of gas dynamics. Arch Ration Mech Anal, 2000, **155**: 277–298

[27] Yu W E, Rykov G, Sinai Ya G. Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. Comm Phys Math, 1996, **177**: 349–380

[28] Elling V. Regular reflection in self-similar potential flow and the sonic criterion. Preprint, 2007

[29] Elling V, Liu T -P. Supersonic flow onto a solid wedge. Comm Pure Appl Math, 2008, **61**: 1347–1448

[30] Huang F -M, Wang Z. Well posedness for pressureless flow. Comm Math Phys, 2001, **222**: 117–146

[31] Gelfand I M. Some problems in the theory of quasilinear equations. Usp Math Nauk, 1959, **14**: 87–158 Amer Math Soc Transl, 1963, **29**: 295–381

[32] Glimm J. Solutions in the large for nonlinear hyperbolic systems of equations. Comm Pure Appl Math, 1965, **18**: 697–715

[33] Glimm J, Ji Xiaomei, Li Jiequan, Li Xiaolin, Zhang Peng, Zhang Tong, Zheng Yuxi. Transonic shock formation in a rarefaction Riemann problem for the 2-D compressible Euler equations preprint. SIAM J Appl Math, 2008, **69**: 720–742

[34] Godunov S K. A difference scheme for numerical solution of discontinuous solution of dydrodynamic equations. Math Sbornik, 1959, **47**: 271–306

[35] Guckenheimer J. Shocks and rarefactions in two space dimensions. Arch Rational Mech Anal, 1975, **59**: 281–291

[36] Kim E H, Song K. Classical solutions for the pressure-gradient equations in non-smooth and non-convex domains. J Math Anal Appl, 2004, **293**(2): 541–550

[37] Kirchinski D J. Solutions of a Riemann problem for a 2×2 system of conservation laws possessing no classical solutions [D]. Adelphi University, 1997

[38] Kofman L, Pogosyan D, Shandarin S. Structure of the universe in the two-dimensional model of adhesion. Mon Nat R Astr Soc, 1990, **242**: 200–208

[39] Kurganov A, Tadmor E. Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers. Numer Methods Partial Differential Equations, 2002, **18**: 584–608

[40] Lax P D. Hyperbolic system of conservation laws II. Comm Pure Appl Math, 1957, **10**: 537–566

[41] Lax P D, Liu X D. Solution of two–dimensional Riemann problem of gas dynamics by positive schemes. SIAM Sci Comp, 1998, **19**: 319–340

[42] Lei Z, Zheng Y -X. A complete global solution to the pressure gradient equation. J Differential Equations, 2007, **236**: 280–292

[43] Levine L E. The expansion of a wedge of gas into a vacuum. Proc Camb Philol Soc, 1968, **64**: 1151–1163

[44] Li J -Q. Note on the compressible Euler equations with zero-temperature. Lett Appl Math, 2001, **14**: 519–523

[45] Li J -Q. On the two-dimensional gas expansion for compressible Euler equations. SIAM J Appl Math, 2001/02, **62**: 831–852

[46] Li J -Q. Global solution of an initial-value problem for two-dimensional compressible Euler equations. J Differential Equations, 2002, **179**: 178–194

[47] Li J -Q, Li W. Riemann problem for the zero-pressure flow in gas dynamics. Progr Natur Sci (English Ed), 2001, **11**: 331–344

[48] Li J -Q, Warnecke G. Generalized characteristics and the uniqueness of entropy solutions to zero-pressure gas dynamics. Adv Differential Equations, 2003, **8**: 961–1004

[49] Li J -Q, Yang H C. Delta-shocks as limits of vanishing viscosity for multidimensional zero-pressure flow in gas dynamics. Q Appl Math, 2001, **59**: 315–342

[50] Li J -Q, Yang Z -C, Zheng Y -X. Characteristic decompositions and interactions of rarefaction waves of 2-D Euler equations. Preprint, 2008

[51] Li J -Q, Zhang T. Generalized Rankine-Hugoniot relations of delta-shocks in solutions of transportation equations. Advances in nonlinear partial differential equations and related areas (Beijing, 1997). River Edge, NJ: World Sci Publ, 1998: 219–232

[52] Li J -Q, Zhang T, Yang S -L. The two-dimensional Riemann problem in gas dynamics//Pitman Monographs and Surveys in Pure and Applied Mathematics 98. Longman, 1998

[53] Li J -Q, Zhang T, Zheng Y -X. Simple waves and a characteristic decomposition of the two dimensional compressible Euler equations. Comm Math Phys, 2006, **267**: 1–12

[54] Li J -Q, Zheng Y -X. Interaction of rarefaction waves of the two-dimensional self-similar Euler equations. Arch Rat Mech Anal, (to appear), 2008

[55] Li J -Q, Zheng Y -X. Interaction of four rarefaction waves in the bi-symmetric class of the two-dimensional Euler equations. submitted for publication, 2008

[56] Li M -J, Zheng Y -X. Semi-hyperbolic patches of solutions of the two-dimensional Euler equations. preprint, 2008

[57] Li Y, Cao Y. Second order “large particle” difference method (in Chinese). Sciences in China (Series A), 1985, **8**: 1024–1035

[58] Liu T -P, Zeng Y -N. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. Mem Amer Math Soc, 1997, **125**(599)

[59] Mach E, Wosyka J. Uber die Fortpflanzungsgeschwindigkeit von Explosionsschallwellen. Sitzungsber Akad Wiss Wien (II Abth), 1875, **72**: 44–52

[60] von Neumann J. Oblique Reflection of Shocks. U S Dept Comm Off of Tech Serv, Washigton, D C PB-37079, 1943

[61] Oleinik O A. Uniqueness and a stability of the generalized solutions of the Cauchy problem for a quasilinear equations. USP Mat Nauk; Amer Math Soc Transl, Ser 2, 1964, **3**: 285–290

[62] Pogodin I A, Suchkov V A, Ianenko N N. On the traveling waves of gas dynamic equations. J Appl Math Mech, 1958, **22**: 256–267

[63] Riemann B. Uber die forpflanzung ebener luftwellen von endlicher schwingungsweite. Abhandl Koenig Gesell Wiss, Goettingen, 1860, **8**: 43

[64] Schulz-Rinne C W. Classification of the Riemann problem for Two-dimensional Riemann problem. SIAM J Math Anal, 1993, **24**: 76–88

[65] Schulz-Rinne C W, Collins J P, Glaz H M. Numerical solution of the Riemann problem for two-dimensional gas dynamics. SIAM J Sci Compt, 1993, **4**: 1394–1414

[66] Shandarin S F, Zeldovich Ya B. The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium. Rev Mod Phys, 1989, **61**: 185–220

[67] Sheng W -C. Two-dimensional Riemann problem for scalar conservation laws. J Differential Equations, 2002, **183**: 239–261

[68] Sheng W -C, Zhang T. The Riemann problem for transportation equations in gas dynamics. Memoirs of AMS, 1997, **137**(654)

[69] Sheng W -C, Zhang T. A cartoon for the climbing ramp problem of a shock and von Neumann paradox. Arch Ration Mech Anal, 2007, **184**: 243–255

[70] Sheng W -C, Yin G. Transonic shock and supersonic shock in the regular reflection of a planar shock. Z Angew Math Phys, DOI 10.1007/s00033-008-8003-4, 2008, (in press)

[71] Song K. The pressure-gradient system on non-smooth domains. Comm Partial Differential Equations, 2003, **28**: 199–221

[72] Song K, Zheng Y -X. Semi-hyperbolic patches of solutions of the pressure gradient system. Disc Cont Dyna Syst, 2009, (inpress)

[73] Suchkov V A. Flow into a vacuum along an oblique wall. J Appl Math Mech, 1963, **27**: 1132–1134

[74] Tan D -C, Zhang T. Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws. I. Four-J cases. J Differential Equations, 1994, **111**(2): 203–254; II. Initial data involving some rarefaction waves. J Differential Equations, 1994, **111**: 255–282

[75] Tan D -C, Zhang T, Zheng Y -X. Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws. J Differential Equations, 1994, **112**: 1–32

[76] Wagner D H. The Riemann problem in two space dimensions for a single conservation law. SIAM J Math Anal, 1983, **14**: 534–559

[77] Wang Z, Huang F -M, Ding X -Q. On the Cauchy problem of transportation equations. Acta Math Appl Sinica (English Ser), 1997, **13**: 113–122

[78] Xin Z -P, Yin H -C. Transonic shock in a nozzle. I. Two-dimensional case. Comm Pure Appl Math, 2005, **58**: 999–1050

[79] Yang H -C, Zhang T. On two-dimensional gas expansion for pressure-gradient equations of Euler system. J Math Anal Appl, 2004, **298**: 523–537

[80] Zakharian A R, Brio, Hunter J K, Webb G M. The von Neumann paradox in weak shock reflection. J Fluid Mech, 2000, **422**: 193–205

[81] Zhang P, Li J -Q, Zhang T. On two-dimensional Riemann problem for pressure-gradient equations of Euler system. Discrete and Continuous Dynamical Systems, 1998, **4**: 609–634

[82] Zhang P, Zhang T. Generalized characteristic analysis and Guckenheimer structure. J Diff Equ, 1999, **52**: 409–430

[83] Zhang T, Zheng Y -X. Two-dimensional Riemann problem for a single conservation law. Trans AMS, 1989, **312**: 589–619

[84] Zhang T, Zheng Y -X. Conjecture on the structure of solution of the Riemann problem for two-dimensional gas dynamics systems. SIAM J Math Anal, 1990, **21**: 593–630

[85] Zhang T, Zheng Y -X. Exact spiral solutions of the two-dimensional Euler equations. Discrete and Continuous Dynamical Systems, 1997, **3**: 117–133

[86] Zhang T, Zheng Y -X. Axisymmetric solutions of the Euler equations for polytropic gases. Arch Rat Mech Anal, 1998, **142**: 253–279

[87] Zheng Y -X. Existence of solutions to the transonic pressure-gradient equations of the compressible Euler equations in elliptic regions. Comm P D Es, 1998, **22**: 1849–1868

[88] Zheng Y -X. Systems of conservation laws. Two-dimensional Riemann problems. Birkhauser, in the series of Progress in Nonlinear Differential Equations, 2001

[89] Zheng Y -X. A global solution to a two-dimensional Riemann problem involving shocks as free boundaries. Acta Math Appl Sin, Engl Ser, 2003, **19**: 559–572

[90] Zheng Y -X. Two-dimensional regular shock reflection for the pressure gradient system of conservation laws. Acta Math Appl Sin, Engl Ser, 2006, **22**: 177–210

[91] Zheng Y -X. The Compressible Euler System in Two Dimensions. Lecture Notes of 2007 Shanghai Mathematics Summer School. International Press and Higher Education Press, (in press)

[92] Zheng Y -X. Absorption of characteristics by sonic curves of the two-dimensional Euler equations. Discrete and Continuous Dynamical Systems, 2009, **23**(1/2): 605–616

[93] Zheng Y -X. Shock reflection for the Euler system//Asakura F (Chief), Aiso H, Kawashima S, Matsumura A, Nishibata S, Nishihara K, ed. Hyperbolic Problems Theory, Numerics and Applications (Proceedings of the Osaka meeting 2004) Vol II. Yokohama Publishers (Japan), 2006: 425–432