Second Order Linear Partial Differential Equations

Part II

Fourier series; Euler-Fourier formulas; Fourier Convergence Theorem; Even and odd functions; Cosine and Sine Series Extensions; Particular solution of the heat conduction equation

Fourier Series

Suppose f is a periodic function with a period $T = 2L$. Then the *Fourier series* representation of f is a trigonometric series (that is, it is an infinite series consists of sine and cosine terms) of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Where the coefficients are given by the *Euler-Fourier formulas*:

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx, \quad m = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

The coefficients a's are called the *Fourier cosine coefficients* (including a_0, the constant term, which is in reality the 0-th cosine term), and b's are called the *Fourier sine coefficients*.

Note 1: Thus, every periodic function can be decomposed into a sum of one or more cosine and/or sine terms of selected frequencies determined solely by that of the original function. Conversely, by superimposing cosines and/or sines of a certain selected set of frequencies we can reconstruct any periodic function.

Note 2: If f is piecewise continuous, then the definite integrals in the Euler-Fourier formulas always exist (i.e. even in the cases where they are improper integrals, the integrals will converge). On the other hand, f needs not to be piecewise continuous to have a Fourier series. It just needs to be periodic. However, if f is not piecewise continuous, then there is no guarantee that we could find its Fourier coefficients, because some of the integrals used to compute them could be improper integrals which are divergent.

Note 3: Even though that the "=" sign is usually used to equate a periodic function and its Fourier series, we need to be a little careful. The function f and its Fourier series "representation" are only equal to each other if, and whenever, f is continuous. Hence, if f is continuous for $-\infty < x < \infty$, then f is exactly equal to its Fourier series; but if f is piecewise continuous, then it disagrees with its Fourier series at every discontinuity. (See the *Fourier Convergence Theorem* below for what happens to the Fourier series at a discontinuity of f.)

Note 4: Recall that a function f is said to be *periodic* if there exists a positive number T, such that $f(x + T) = f(x)$, for all x in its domain. In such a case the number T is called a *period* of f. A period is not unique, since if $f(x + T) = f(x)$, then $f(x + 2T) = f(x)$ and $f(x + 3T) = f(x)$ and so on. That is, every integer-multiple of a period is again another period. The smallest such T is called the *fundamental period* of the given function f. A special case is the constant functions. Every constant function is clearly a periodic function, with an arbitrary period. It, however, has no fundamental period, because its period can be an arbitrarily small real number. The Fourier series representation defined above is unique for each function with a fixed period $T = 2L$. However, since a periodic function has infinitely many (non-fundamental) periods, it can have many different Fourier series by using different values of L in the definition above. The difference, however, is really in a technical sense. After simplification they would look the same.

Therefore, technically at least, a Fourier series of a periodic function depends both on the function as well as its chosen period.

Note 5: The definite integrals in the Euler-Fourier formulas can be found be integrating over *any* interval of length $2L$. However, from $-L$ to L is the convention, and is often the most convenient interval to use.

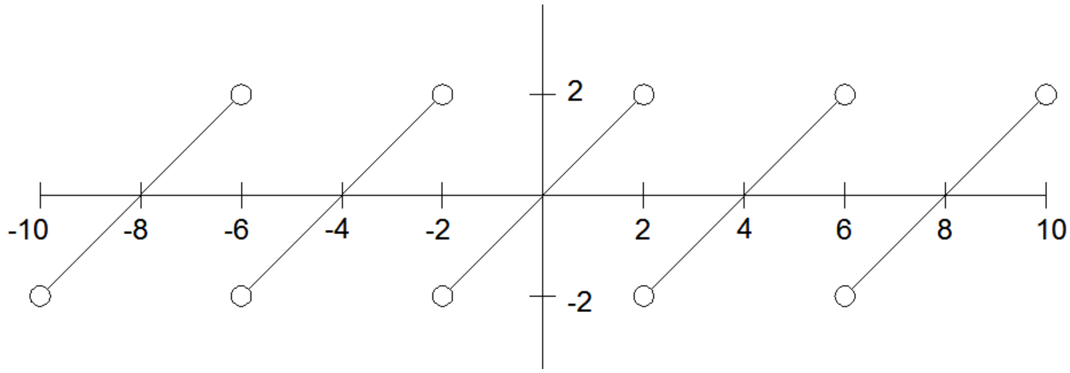
Note 6: Since the Fourier coefficients are calculated by definite integrals, which are insensitive to the value of the function at finitely many points. Consequently, piecewise continuous functions of the same period that differ from each other at finitely many points (notably, at isolated discontinuities) per period will have the same Fourier series.

Note 7: The constant term in the Fourier series, which has expression

$$\frac{a_0}{2} = \frac{1}{2} \cdot \frac{1}{L} \int_{-L}^{L} f(x) \cos(0)\, dx = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx ,$$

is just the *average* or *mean value* of $f(x)$ on the interval $[-L, L]$. Since f is periodic, this average value is the same for every period of f. Therefore, the constant term in a Fourier series represents the average value of the function f over its entire domain.

Example: Find a Fourier series for $f(x) = x, \ -2 < x < 2, f(x + 4) = f(x)$.



First note that $T = 2L = 4$, hence $L = 2$.
The constant term is one half of:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_{-2}^{2} = \frac{1}{2}(2-2) = 0$$

The rest of the cosine coefficients, for $n = 1, 2, 3, \ldots$, are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\left.\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\right|_{-2}^{2} - \frac{2}{n\pi}\int_{-2}^{2}\sin\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\left(\left.\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left(\left(0 + \frac{4}{n^2\pi^2}\cos(n\pi)\right) - \left(0 + \frac{4}{n^2\pi^2}\cos(-n\pi)\right)\right) = 0$$

Hence, there is no nonzero cosine coefficient for this function. That is, its Fourier series contains no cosine terms at all. (We shall see the significance of this fact a little later.)

The sine coefficients, for $n = 1, 2, 3, \ldots$, are

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2}\bigg|_{-2}^{2} - \frac{-2}{n\pi}\int_{-2}^{2}\cos\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\bigg|_{-2}^{2}\right)$$

$$= \frac{1}{2}\left(\left(\frac{-4}{n\pi}\cos(n\pi) - 0\right) - \left(\frac{4}{n\pi}\cos(-n\pi) - 0\right)\right)$$

$$= \frac{-2}{n\pi}\left(\cos(n\pi) + \cos(n\pi)\right) = \frac{-4}{n\pi}\cos(n\pi)$$

$$= \begin{cases} \dfrac{4}{n\pi}, & n = odd \\ \dfrac{-4}{n\pi}, & n = even \end{cases} \qquad = \frac{(-1)^{n+1}4}{n\pi}.$$

Therefore, $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$.

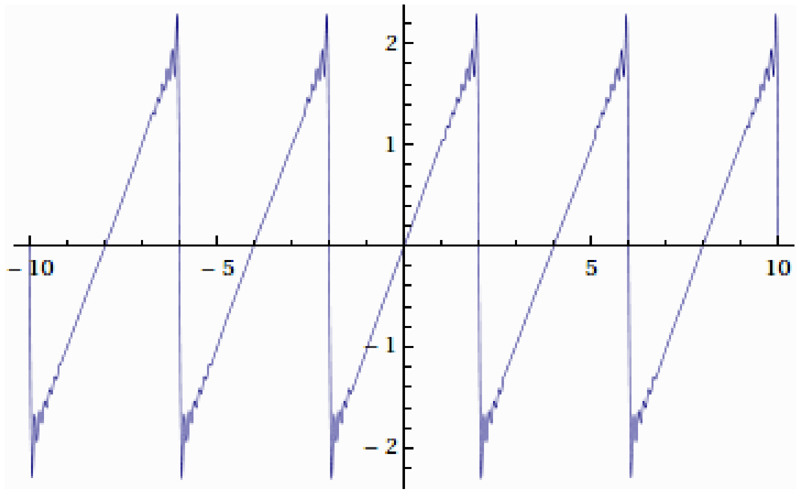


Figure: the graph of the partial sum of the first 30 terms of the Fourier series

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}.$$

Compare it against the graph of the actual function the series represents the function $f(x) = x$, $-2 < x < 2$, $f(x + 4) = f(x)$, seen earlier.

Example: Find a Fourier series for $f(x) = x$, $0 < x < 4$, $f(x + 4) = f(x)$. How will it be different from the series above?

$$a_0 = \frac{1}{2}\int_0^4 x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^4 = \frac{1}{2}(8-0) = 4$$

For $n = 1, 2, 3, \ldots$:

$$a_n = \frac{1}{2}\int_0^4 x\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left(\left(0 + \frac{4}{n^2\pi^2}\cos(2n\pi)\right) - \left(0 + \frac{4}{n^2\pi^2}\cos(0)\right)\right) = 0$$

$$b_n = \frac{1}{2}\int_0^4 x\sin\frac{n\pi x}{2}\,dx$$

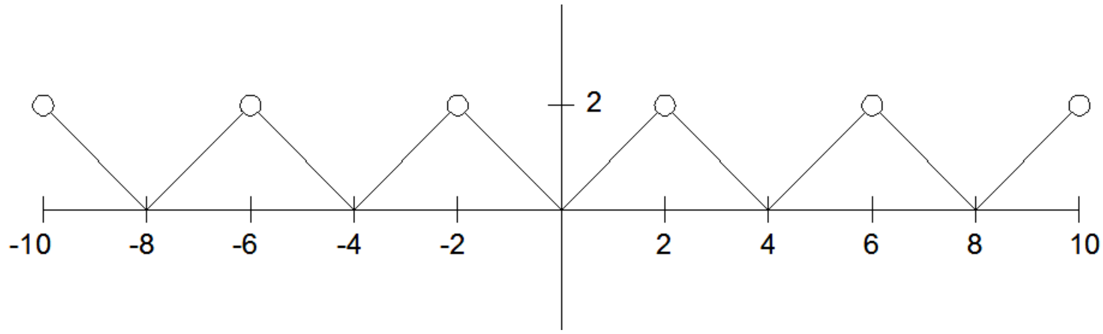
$$= \frac{1}{2}\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_0^4\right)$$

$$= \frac{1}{2}\left(\left(\frac{-8}{n\pi}\cos(2n\pi) - 0\right) - (0-0)\right) = \frac{-4}{n\pi}$$

Consequently,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = 2 + \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}\ .$$

Example: Find a Fourier series for $f(x) = |x|$, $-2 < x < 2$, $f(x + 4) = f(x)$.



$$\textit{Answer}: \quad f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}$$

Example: Find a Fourier series for

$$f(x) = \begin{cases} -2, & -1 \le x < 0 \\ 2, & 0 \le x < 1 \end{cases}, \qquad f(x + 2) = f(x).$$

$$\textit{Answer}: \quad f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin((2n-1)\pi x)$$

Comment: Just because a Fourier series could have infinitely many (nonzero) terms does not mean that it will always have that many terms. If a periodic function f can be expressed by finitely many terms normally found in a Fourier series, then the expression must be the Fourier series of f. (This is analogous to the fact that the Maclaurin series of any polynomial function is just the polynomial itself, which is a sum of finitely many powers of x.)

Example: The Fourier series (period 2π) representing $f(x) = 5 + \cos(4x) - \sin(5x)$ is just $f(x) = 5 + \cos(4x) - \sin(5x)$.

Example: The Fourier series (period 2π) representing $f(x) = 6\cos(x)\sin(x)$ is not exactly itself as given, since the product $\cos(x)\sin(x)$ is not a term in a Fourier series representation. However, we can use the double-angle formula of sine to obtain the result: $6\cos(x)\sin(x) = 3\sin(2x)$. Consequently, the Fourier series is $f(x) = 3\sin(2x)$.

The Fourier Convergence Theorem

Here is a theorem that states a sufficient condition for the convergence of a given Fourier series. It also tells us to what value does the Fourier series converge to at each point on the real line.

Theorem: Suppose f and f' are piecewise continuous on the interval $-L \le x \le L$. Further, suppose that f is defined elsewhere so that it is periodic with period $2L$. Then f has a Fourier series as stated previously whose coefficients are given by the Euler-Fourier formulas. The Fourier series converge to $f(x)$ at all points where f is continuous, and to

$$\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right] / 2$$

at every point c where f is discontinuous.

Comment: As seen before, the fact that f is piecewise continuous guarantees that the Fourier coefficients can be found. The condition that f' is also piecewise continuous is a sufficient condition to guarantee that the series thusly found will be convergent everywhere on the real line. As well, recall that, suppose f is continuous at c, then by definition $f(c)$ equals both one-sided limits of $f(x)$ as x approaches c. Therefore, the second part of the theorem could be even more succinctly stated as that the Fourier series representing f will always converge to

$$\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right] / 2$$

at every point c (and not just at discontinuities of f).

A consequence of this theorem is that the Fourier series of f will "fill in" any removable discontinuity the original function might have. A Fourier series will not have any removable-type discontinuity.

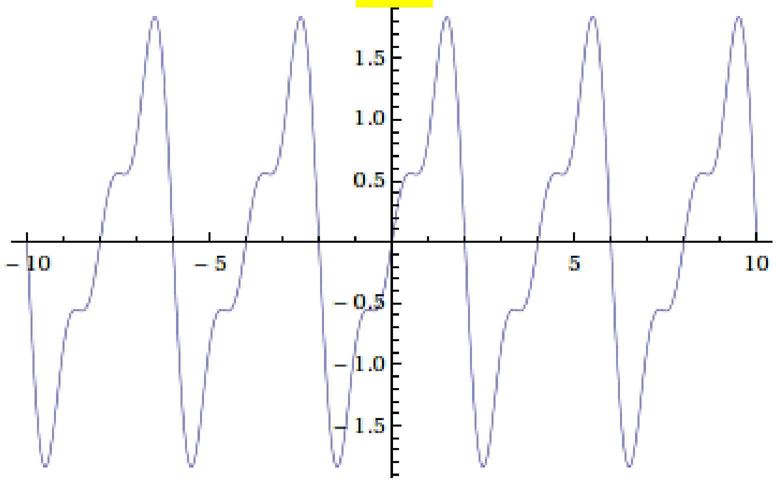
Example: Let us revisit the earlier calculation of the Fourier series representing $f(x) = x, \quad -2 < x < 2, \qquad f(x + 4) = f(x)$.

The Fourier series, as we have found, is

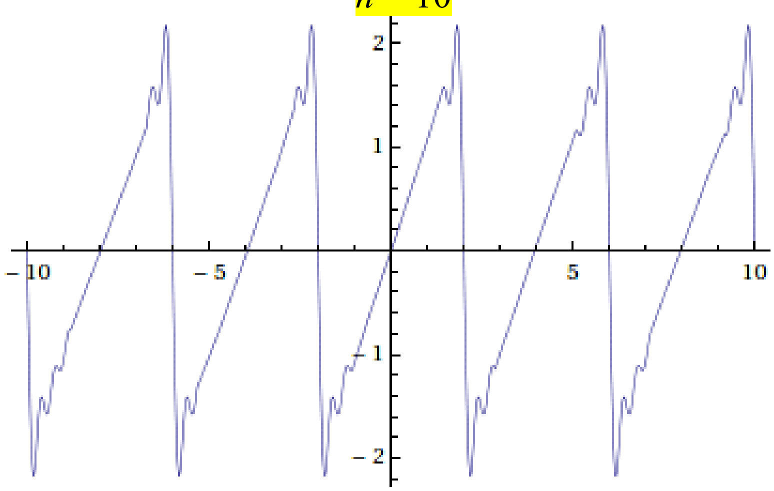
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}.$$

The following figures are the graphs of various finite *n*-th partial sums of the series above.

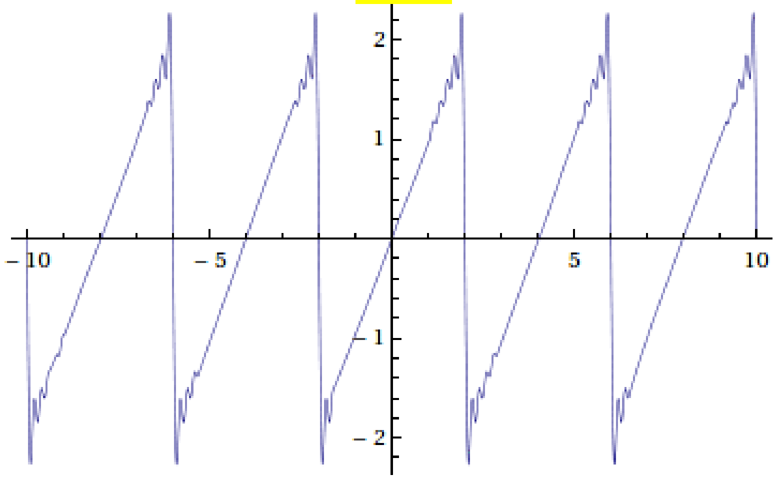
$n = 3$



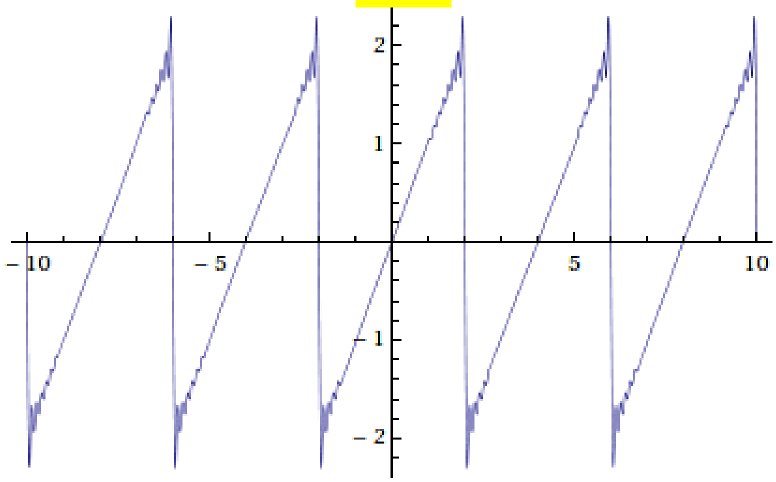
$n = 10$



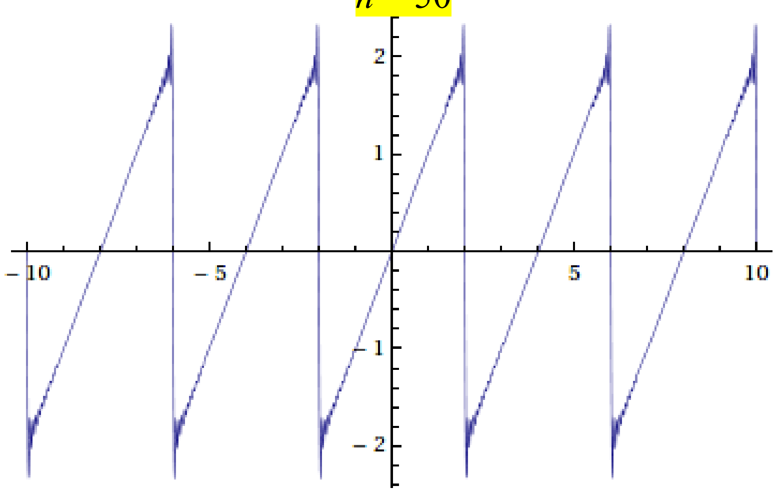
$n = 20$



$n = 30$



$n = 50$



Note that superimposed sinusoidal curves take on the general shape of the piecewise continuous periodic function $f(x)$ almost immediately. As well, for the parts of the curve where $f(x)$ is continuous (where the Fourier Convergence Theorem predicts a perfect match) the composite curve of the Fourier series converges rapidly to that of $f(x)$, as predicted. The convergence is not as rapidly near the jump discontinuities. Indeed, for all but the lowest partial sums of the Fourier series, the curve seems to "overshoot" that of $f(x)$ near each jump discontinuity by a noticeable margin. Further more, this discrepancy does not fade away for any finitely larger n. That is, the convergence of a Fourier series, while predictable, is *not uniform*. (That is a small price we pay for approximating a piecewise continuous periodic function by sinusoidal curves. It can be done, but the Fourier series does not converge uniformly to the actual function.)

This behavior is known as the *Gibbs Phenomenon*. It further states that the partial sums of a Fourier series will overshoot a jump discontinuity by an amount approximately equal to 9% of the jump. That is, near each jump discontinuity, the overshoot amounts to about

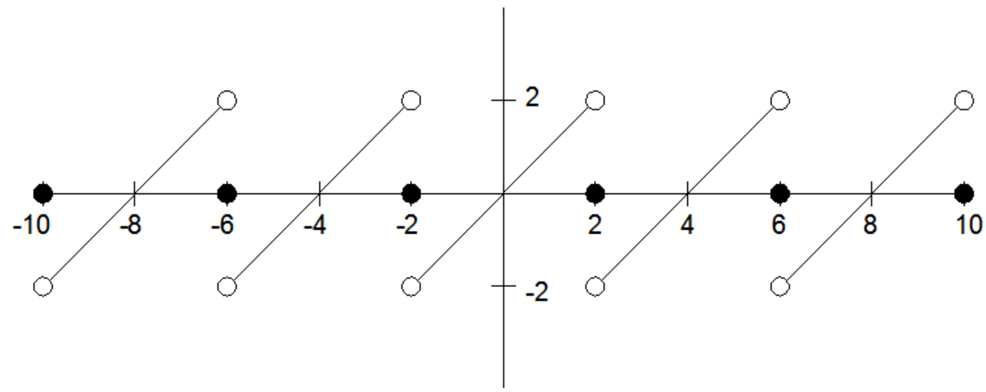
$$0.09\left| \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x) \right|,$$

for large n. Further, this overshoot does not go away for any finitely large n.

Question: Sketch the graph of the Fourier series of

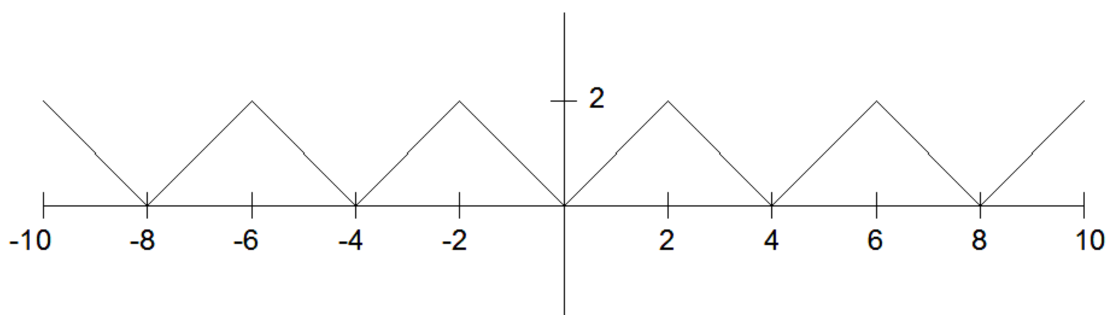
$$f(x) = x, \qquad -2 < x < 2, \quad f(x+4) = f(x).$$

We have seen a few graphs of its partial sums. But what will the graph of the actual Fourier series look like?



Example: Sketch the graph of the Fourier series of

$$f(x) = |x|, -2 < x < 2, f(x+4) = f(x).$$



Example: Sketch the graph of the Fourier series of

$$f(x) = \begin{cases} -2, & -1 \le x < 0 \\ 2, & 0 \le x < 1 \end{cases}, \qquad f(x+2) = f(x).$$

Even and Odd Functions

Recall that an <u>even</u> function is any function f such that

$$f(-x) = f(x), \qquad \text{for all } x \text{ in its domain.}$$

Examples: cos(x), sec(x), any constant function, x^2, x^4, x^6, … , x^{-2}, x^{-4}, …

An <u>odd</u> function is any function f such that

$$f(-x) = -f(x), \qquad \text{for all } x \text{ in its domain.}$$

Examples: sin(x), tan(x), csc(x), cot(x), x, x^3, x^5, … , x^{-1}, x^{-3}, …

Most functions, however, are neither even nor odd. There is one function that is both even and odd. (What is it?)

Arithmetic Combinations of Even and Odd Functions

The table below summaries the result of performing the common arithmetic operations on a pair of even and/or odd functions:

	Even and Even	Odd and Odd	Even and Odd
+ / −	Even	Odd	Neither
× / ÷	Even	Even	Odd

The result above can be extended to arbitrarily many terms. For example, a sum of three or more even functions will again be even. (Care needs to be taken in the cases where 3 or more odd functions forming a product/quotient. For example, a product of 3 odd functions will be odd, but a product of 4 odd functions is even.)

Calculus Properties of Even and Odd Functions

Suppose f is an even function, continuous on $-L \le x \le L$, then $$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx .$$
Suppose f is an odd function, continuous on $-L \le x \le L$, then $$\int_{-L}^{L} f(x)\,dx = 0 .$$

The Fourier Cosine Series

Suppose f is an even periodic function of period $2L$, then its Fourier series contains only cosine (include, possibly, the constant term) terms. It will not have any sine term. That is, its Fourier series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} .$$

Conversely, any periodic function whose Fourier series has the form of a cosine series as shown must be an even periodic function. Computationally, this means that the Fourier coefficients of an even periodic function are given by

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{m\pi x}{L}\,dx, \qquad m = 0, 1, 2, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

Notice that the integrand in the definite integral used to find the cosine coefficients a's is an even function (it is a product of two even functions, $f(x)$ and $\cos x$). Therefore, we can use the symmetric property of even functions to simplify the integral.

The Fourier Sine Series

If f is an odd periodic function of period $2L$, then its Fourier series contains only sine terms. It will not have any cosine term. That is, its Fourier series is of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Conversely, any periodic function whose Fourier series has the form of a sine series as shown must be an odd periodic function. Therefore, the Fourier coefficients of an odd periodic function are given by

$$a_m = 0, \qquad m = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx\,, \qquad n = 1, 2, 3, \ldots$$

Example: We have calculated earlier that the function $f(x) = x$, $-2 < x < 2$, $f(x + 4) = f(x)$, has as its Fourier series consists of purely sine terms:

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}\,.$$

We now see that this sine series signifies that the function is odd periodic.

It is perhaps not very obvious, but the integrand in the integral for the Fourier sine coefficients is another even function. It is a product of two odd functions, $f(x)$ and $\sin x$, which makes it even. Therefore, we can again take advantage of the symmetric property of even functions to simplify the integral.

The Cosine and Sine Series Extensions

If f and f' are piecewise continuous functions defined on the interval $0 \le t \le L$, then f can be extended into an even periodic function, F, of period $2L$, such that $f(x) = F(x)$ on the interval $[0, L]$, and whose Fourier series is, therefore, a cosine series. Similarly, f can be extended into an odd periodic function of period $2L$, such that $f(x) = F(x)$ on the interval $(0, L)$, and whose Fourier series is, therefore, a sine series. The process that such extensions are obtained is often called cosine /sine series *half-range expansions*.

Here is an outline of how this can be done. Start with a function that is defined only on an interval of finite length, from 0 to L. First expand the function to be defined on the interval from $-L$ to L such that the function is an even or an odd function as required. Then define the function to be periodic with a period of $T = 2L$ by requiring $F(x + 2L) = F(x)$. This process is actually much easier than it sounds. Mathematically, the process can be achieved rather simply, as described below.

Even (cosine series) extension of $f(x)$

Given $f(x)$ defined on $[0, L]$. Its even extension of period $2L$ is:

$$F(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L < x < 0 \end{cases}, \qquad F(x + 2L) = F(x).$$

Where $$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},$$ such that

$$a_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx, \qquad m = 0, 1, 2, 3, \ldots$$

$$b_n = 0, \qquad n = 1, 2, 3, \ldots$$

Odd (sine series) extension of $f(x)$

Given $f(x)$ defined on $(0, L)$. Its odd extension of period $2L$ is:

$$F(x)=\begin{cases} f(x), & 0<x<L \\ 0, & x=0, L \\ -f(-x), & -L<x<0 \end{cases}, \qquad F(x+2L)=F(x).$$

Where $$F(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$ such that

$a_m = 0,$ $\qquad m = 0, 1, 2, 3, \ldots$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots$$

Example: Let $f(x) = x$, $0 \le x < 2$. Find its cosine and sine series extensions of period 4.

Answers: Cosine series: $$f(x)=1-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}$$

Sine series: $$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

Back to the Heat Conduction Problem

Previously, we had found the general solution of the initial-boundary value problem given by the one-dimensional heat conduction equation modeling a bar that has both of its ends kept at 0 degree. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L} .$$

Setting $t = 0$ and applying the initial condition $u(x, 0) = f(x)$, we get

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x) .$$

We now know that the above equation says that the initial condition needs to be an odd periodic function of period $2L$. Since the initial condition could be an arbitrary function, it usually means that we would need to "force the issue" and expand it into an odd periodic function of period $2L$. That is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} .$$

Compare the two expressions, we see that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} .$$

Therefore, the particular solution is found by setting all the coefficients $C_n = b_n$, where b_n's are the Fourier sine coefficients of (or the odd periodic extension of) the initial condition $f(x)$:

$$C_n = b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx .$$

Example: Solve the heat conduction problem

$$8\,u_{xx} = u_t \quad , \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } u(5, t) = 0,$$
$$u(x, 0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x).$$

Since the standard form of the heat conduction equation is $\alpha^2 u_{xx} = u_t$, we see that $\alpha^2 = 8$; and we also note that $L = 5$. Therefore, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} C_n\, e^{-8n^2 \pi^2 t/25} \sin\frac{n\pi x}{5}$$

The initial condition, $f(x)$, is already an odd periodic function (notice that it is a Fourier sine series) of the correct period $T = 2L = 10$. Therefore, no additional calculation is needed, and all we need to do is to extract the correct Fourier sine coefficients from $f(x)$. To wit

$C_5 = b_5 = 2$,
$C_{10} = b_{10} = -4$,
$C_{25} = b_{25} = 1$,
$C_n = b_n = 0$, for all other n, $n \neq 5, 10$, or 25.

Hence,

$$u(x,t) = 2e^{-8(5^2)\pi^2 t/25} \sin(\pi x) - 4e^{-8(10^2)\pi^2 t/25} \sin(2\pi x)$$
$$+\, e^{-8(25^2)\pi^2 t/25} \sin(5\pi x)$$

What will the particular solution be if the initial condition is $u(x, 0) = x$ instead? That is, solve the following heat conduction problem:

$$8\,u_{xx} = u_t \quad , \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \text{ and } u(5, t) = 0,$$
$$u(x, 0) = x.$$

The general solution is still

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-8n^2\pi^2 t/25} \sin\frac{n\pi\, x}{5} .$$

The initial condition is an odd function, but it is not a periodic function. Therefore, it needs to be expanded into its odd periodic extension of period 10 ($T = 2L$). Its coefficients are, for $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi\, x}{L}\,dx = \frac{2}{5}\int_0^5 x\sin\frac{n\pi\, x}{5}\,dx$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5}\Bigg|_0^5 - \frac{-5}{n\pi}\int_0^5 \cos\frac{n\pi x}{5}\,dx\right)$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5} + \frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}\Bigg|_0^5\right)$$

$$= \frac{2}{5}\left(\left(\frac{-25}{n\pi}\cos(n\pi) - 0\right) - (0 - 0)\right)$$

$$= \frac{-10}{n\pi}\cos(n\pi)$$

$$= \begin{cases} \dfrac{10}{n\pi}, & n = odd \\ \dfrac{-10}{n\pi}, & n = even \end{cases} \qquad = \frac{(-1)^{n+1}10}{n\pi}$$

The resulting sine series is (representing the function $f(x) = x$, $-5 < x < 5$, $f(x + 10) = f(x)$):

$$f(x) = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{5}.$$

The particular solution can then be found by setting each coefficient, C_n, to be the corresponding Fourier sine coefficient of the series above, $C_n = b_n = \dfrac{(-1)^{n+1} 10}{n\pi}$. Therefore, the particular solution is

$$u(x,t) = \frac{10}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-8n^2\pi^2 t/25} \sin \frac{n\pi x}{5}.$$

Exercises E-2.1:

1 – 8 Find the Fourier series representation of each periodic function. Determine the values to which each series converge to at $x = 0$.

1. $f(x) = 5, \quad -1 < x < 1, \quad f(x + 2) = f(x).$
2. $f(x) = (\cos x + \sin x)^2, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x).$
3. $f(x) = 6 - 3x, \quad 0 \le x < 2, \quad f(x + 2) = f(x).$
4. $f(x) = x^2, \quad 0 \le x < \pi, \quad f(x + \pi) = f(x).$
5. $f(x) = x^2, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x).$
6. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 4, & 0 \le x \le 1 \\ 0, & 1 < x < 2 \end{cases}, \quad f(x + 4) = f(x).$
7. $f(x) = |\sin x|, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x).$
8. $f(x) = \delta(x - c), \quad -\pi < x < \pi, \quad 0 < c < \pi, \quad f(x + 2\pi) = f(x).$

9 – 12 Expand each function into its cosine series and sine series representations of the indicated period. Determine the values to which each series converge to at $x = 0$, $x = 2$, and $x = -2$.

9. $f(x) = 3 - x, \quad T = 6.$
10. $f(x) = e, \quad T = 2\pi.$
11. $f(x) = \sin x, \quad T = 2\pi.$
12. $f(x) = \begin{cases} x, & 0 \le x < 2 \\ 2, & 2 \le x < 3 \end{cases}, \quad T = 6.$

13. Solve the heat conduction problem

$$2u_{xx} = u_t, \quad 0 < x < 9, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(9, t) = 0,$$
$$u(x, 0) = 25\sin(\pi x/3) + 45\sin(4\pi x/3) - 12\sin(3\pi x).$$

14. Solve the heat conduction problem of the given initial conditions.

$$9u_{xx} = u_t, \quad 0 < x < 12, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(12, t) = 0,$$

(a) $u(x, 0) = 3\sin(\pi x) - \sin(7\pi x/6) - 6\sin(2\pi x),$
(b) $u(x, 0) = 4,$
(c) $u(x, 0) = 4 - x.$

Answers E-2.1:

1. $f(x) = 5$, $f(0) = 5$.
2. $f(x) = 1 + \sin(2x)$, $f(0) = 1$.
3. $f(x) = 3 + \frac{6}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)$, $f(0) = 3$.
4. $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(2nx) - \frac{\pi}{n}\sin(2nx)\right]$, $f(0) = \frac{\pi^2}{2}$.
5. $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$, $f(0) = 0$.
6. $f(x) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{n}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{2}\right)\right]$,
   $f(0) = 2$.
7. $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\cos(2nx)$, $f(0) = 0$.
8. $f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n(x-c))$, $f(0) = 0$.
9. Cosine series: $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{12}{(2n-1)^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)$,

   Sine series: $f(x) = \sum_{n=1}^{\infty}\frac{9}{n\pi}\sin\left(\frac{n\pi x}{3}\right)$;

The cosine series converges to 3, 1, and 1; the sine series converges to 0, 1, and −1, respectively, at $x = 0$, $x = 2$, and $x = -2$.

10. Cosine series: $f(x) = e$,

   Sine series: $f(x) = \sum_{n=1}^{\infty}\frac{4e}{(2n-1)\pi}\sin(nx)$;

   The cosine series converges to e at all 3 points. The sine series converges to 0, e, and $-e$, respectively, at $x = 0$, $x = 2$, and $x = -2$.

11. Cosine series: $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}\cos(2nx)$,

   Sine series: $f(x) = \sin x$;

   Both series converge to 0 at $x = 0$ and to sin(2) at $x = 2$. At $x = -2$, the cosine series converges to sin(2), the sine series to −sin(2).

12. Cosine series: $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{6}{n^2\pi^2} \left(\cos\left(\frac{2n\pi}{3}\right) - 1 \right) \cos\left(\frac{n\pi x}{3}\right)$,

Sine series: $f(x) = \sum_{n=1}^{\infty} \left[\frac{6}{n^2\pi^2} \sin\left(\frac{2n\pi}{3}\right) - \frac{4(-1)^n}{n\pi} \right] \sin\left(\frac{n\pi x}{3}\right)$;

Both series converge to 0 at $x = 0$ and to 2 at $x = 2$. At $x = -2$, the cosine series converges to 2, the sine series to -2.

13. $u(x,t) = 25e^{-2\pi^2 t/9} \sin(\frac{\pi x}{3}) + 45e^{-32\pi^2 t/9} \sin(\frac{4\pi x}{3}) - 12e^{-18\pi^2 t} \sin(3\pi x)$

14. (a) $u(x,t) = 3e^{-9\pi^2 t} \sin(\pi x) - e^{-49\pi^2 t/4} \sin(\frac{7\pi x}{6}) - 6e^{-36\pi^2 t} \sin(2\pi x)$;

(b) $u(x,t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-9n^2\pi^2 t/144} \sin\frac{n\pi x}{12}$.

