Dragon curves revisited

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It has happened several times in recent history that a mathematical discovery of great beauty and importance was originally published in a journal that would not likely to be read by many a working mathematician.

One famous example is the Penrose tilings [15]. Surely, Penrose tilings and the theory of quasicrystals is now a major area of research (see, e.g., [17]), and not only in mathematics but also in physics and chemistry, as witnessed by the 2011 Nobel Prize awarded to D. Shechtman for "the discovery of quasicrystals" in 1982. It is a pleasure to mention that this magazine played a role in popularizing Penrose tilings [16].

The topic of this column is another mathematical object of comparable beauty, the Dragon curves, whose theory was created by Chandler Davis and Donald Knuth [4]. The original articles are not easily available (they are reprinted in [10], along with previously unpublished addendum).[1]

Mathematical Intelligencer wrote about Dragon curves more than 30 years ago [5, 6, 7]. In spite of the existence of a Wikipedia article on the subject and in spite of their appearance in M. Crichton's popular novel "Jurassic Park", Dragon curves are not sufficiently well known to contemporary mathematicians, especially the younger ones who missed the original excitement of 40+ years ago.

The goal of this article is to bring Dragon curves to spotlight again and to pay tribute to Chandler Davis, a co-author of an elegant theory that explains the striking features of these curves. This article is merely an invitation to the subject; the reader should not expect a thorough survey of the results or proofs.

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[1]It is worth mentioning that neither [15] nor [4] can be found on MathSciNet. Not surprisingly, it was Martin Gardner who popularized Penrose tilings and Dragon curves in his Scientific American column, in 1977 and 1967, respectively.

The Dragon curve was discovered (or shall one say, invented) by a NASA physicist John Heighway in 1966 and named by his colleague William Harter. Here is the story as told by Harter, reproduced from [10]:

> The dragon curve was born in June 1966. Jack [Heighway] came into my office (actually cubicle) and said that if you folded a $1 bill repeatedly he thought it would make a random walk or something like that. (We'd been arguing about something in Feller's book on return chances.) I was dubious but said "Let's check it out with a big piece of paper." (Those were the days when NASA could easily afford more than $1's worth of paper.) Well, it made a funny pattern alright but we couldn't really see it too clearly.[2] So one of us thought to use tracing paper and "unfold" it indefinitely so we could record (tediously) as big a pattern as we wanted. But each time we made the next order, it just begged us to make one more!

So, take a strip of paper and fold it in half, then in half again, several times. Now unfold the paper: you see a sequence of creases that are labeled D and U, for down and up, see Figure 1.

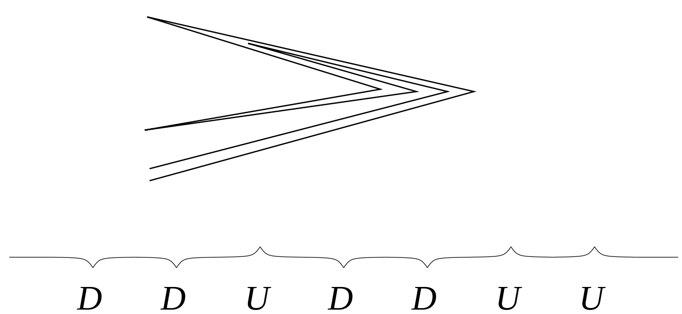

Figure 1: Folding a strip of paper three times.

The result of n foldings is a sequence S_n of letters D and U of length $2^n - 1$. There are two inductive rules describing S_{n+1} via S_n. Given a sequence S,

[2] It is a common belief that the maximal number of times any piece of paper could be folded in half is seven. This is not so: apparently, the current world record belongs to Britney Gallivan who, when a high school student in 2002, managed to fold a single 4000 ft long piece of toilet paper in half twelve times [9].

let $\bar{S}$ be the same sequence, read from right to left and with the letters D and U swapped. For example, $\overline{DDU} = DUU$. Then one has

$$S_{n+1} = S_n D \overline{S}_n. \qquad (1)$$

The reason is that folding $n+1$ times is achieved by folding once (letter D in the middle of S_{n+1}), followed by folding n times (the string S_n at the beginning of S_{n+1} and its reverse at the end). Since S_{n+1} starts with S_n, the limiting infinite sequence S_∞ is well defined.

The other way to obtain S_{n+1} from S_n is as follows. Let $S_n = a_1 a_2 \dots a_m$ where $m = 2^n - 1$ and each a_i is either D or U. Then

$$S_{n+1} = D a_1 U a_2 D a_3 U \dots D a_m U. \qquad (2)$$

The reason is that folding $n+1$ times is also achieved by folding n times, and then once. Therefore the 2nd, 4th, etc., creases of S_{n+1} are the same as those of S_n, whereas the 1st, 3rd, 5th, etc., creases are $D, U, D, U, \dots$ in alternating order.

Now open the strip of paper so that every crease makes the right angle, and round the angles slightly. One obtains a Dragon curve, see Figure 2. Letter D is interpreted as the left turn and letter U as the right one.

One may use other angles when opening the strip, see Figure 3.

The two recursion rules (1) and (2) have geometric interpretations.

Let Γ_n be the Dragon curve of nth generation and let O be its end point. Turn Γ_n about O through 90° and attach this new curve to Γ_n to obtain Γ_{n+1}. The reader will convince herself that this is a reformulation of rule (1).

The geometric interpretation of rule (2) consists of considering each segment of Γ_n as the hypothenuse of a right isosceles triangle and replacing it by the two catheti, alternating between right and left side, see Figure 4. The resulting curve has twice as many segments and is encoded by the word S_{n+1} as in (2).

Figure 4 suggests rescaling of each next generation by the factor $1/\sqrt{2}$: this rescaling keeps the size of the curves Γ_n fixed. There is a natural limit (in the Hausdorff metric), as $n \to \infty$, of the curves Γ_n. We call this limiting curve the Dragon and denote it by Γ_∞.

The Davis-Knuth theory makes it possible to analyze the Dragon curve in detail. Denote by $g(n)$ the excess of Ds over Us among the first $n-1$ letters of the infinite word S_∞. The sequence $g(n)$ starts as follows:

$$0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2, 1, \dots$$

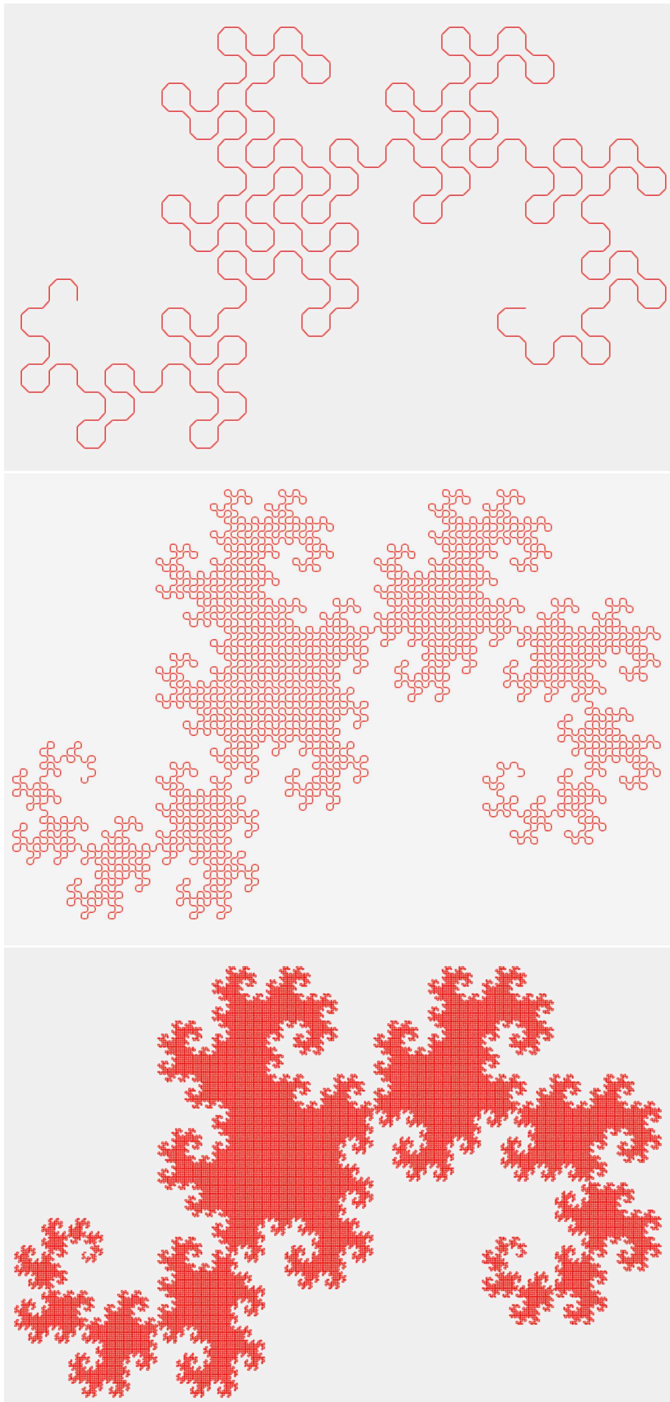

Figure 2: Dragon curves of 8th, 12th and 16th generations.

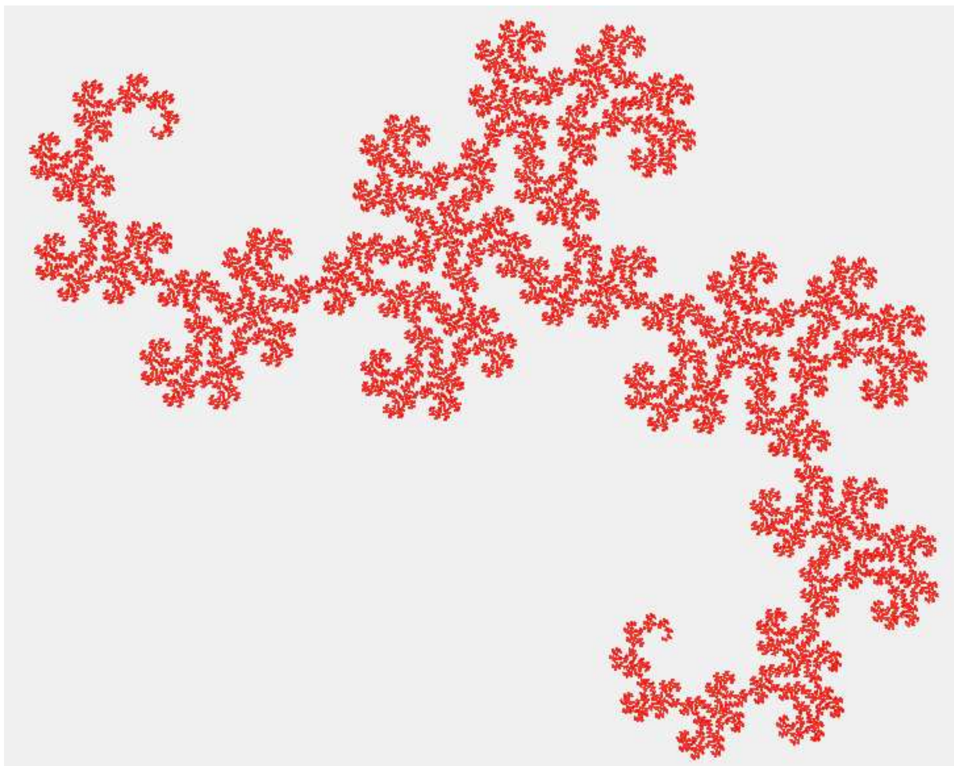

Figure 3: Dragon curve with the opening angle $17\pi/32$.

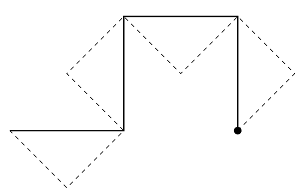

Figure 4: Geometric interpretation of rule (2): Γ_n is in solid line and Γ_{n+1} is in dashed one.

This is sequence A005811 in Sloan's OEIS. It has the property that $g(2^k) = 1$ and satisfies the recurrence

$$g(2^{k+1} + 1 - m) = 1 + g(m)$$

for $1 \leq m \leq 2^k$. In particular, all terms are positive.

One can compute the coordinates of the vertices of a Dragon curve in terms of this sequence. Let $\omega = e^{i\theta}$ and suppose that the opening angle of the strip of paper is $\pi - \theta$. Assume that the segments of the Dragon curve are of unit length and the first one goes from 0 to 1. Then the complex number representing the nth vertex is

$$V_n = \omega^{g(1)} + \omega^{g(2)} + \ldots + \omega^{g(n)}.$$

In fact, one can compute this complex number explicitly using a special number representation.

Theorem 1 *Let*

$$n = 2^{k_0} - 2^{k_1} + \ldots + (-1)^t 2^{k_t} \quad \text{with} \quad k_0 > k_1 > \ldots > k_t \geq 0.$$

Then

$$V_n = (1+\omega)^{k_0} - \omega(1+\omega)^{k_1} + \ldots + (-\omega)^t (1+\omega)^{k_t}.$$

For example, if $\omega = 1$ then $V_n = n$, as it should be: the strip of paper is laid out straight. The most interesting right angle turn corresponds to $\omega = \sqrt{-1}$.

Assume that the turning angle is 90°. Perhaps the most striking property of the Dragon curve is next result of the Davis-Knuth theory.

Theorem 2 *The Dragon curve does not cross itself (so its rounded version is embedded). Four copies of the Dragon curve, starting at the same point and rotated* 90°*, fill the plane: each segment of the standard grid is traversed exactly once; see Figure 5.*

Theorem 2 implies that the dimension of Γ_∞ is two, as indeed suggested by Figure 2. This figure also reveals self-similarity of the Dragon. It is proved in [14] that the Dragon consists of a countable union of geometrically similar disk-like sets that intersect each other at single points in linear order. The Hausdorff dimension of the boundary of the Dragon is computed in [3]: its numerical value is approximately 1.523627.

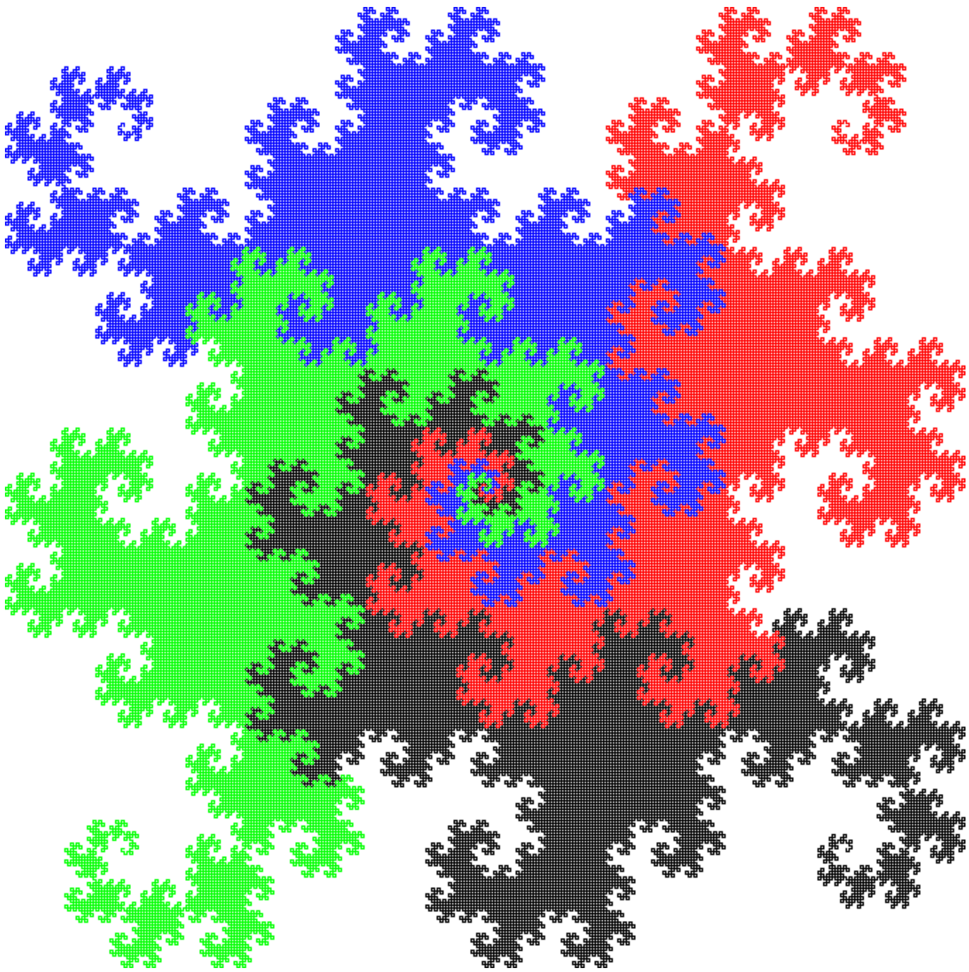

Figure 5: Four copies of the Dragon curve fit together tightly. Image due to S. Lew, Wikimedia Commons.

It is worth mentioning that when the opening angle θ reaches $60°$, a kind of phase transition occurs. If the angle is less than $60°$ then the respective Dragon curves of all generations are bounded. If $\theta = 60°$ then the Dragon curve "lives" on the hexagonal grid and expands linearly with the number of generation n.[3]

Of course, there is more than one way to fold a strip of paper. Previously all folds were in the same direction but, in fact, one has two choices for each new fold, so there are 2^n combinatorial patterns of n folds leading to a wide variety of such generalized Dragon curves. For example, one can change the direction every time, leading to the sequence

$$S_1 = D, S_2 = DUU, S_3 = DUUDDDU, S_4 = DUUDDDUUDUUUDDU, \ldots$$

The respective Dragon curve fills a quarter of the plane, see Figure 6. The boundary of this Dragon curve is not very interesting but its fine structure is quite intricate.

Appropriate versions of recurrences (1) and (2) hold for generalized Dragon curves. For the first rule, one has a choice of whether to insert letter D or U in the middle at nth step. For the second, one has a choice of whether to start attaching the right isosceles triangles on the right or on the left of the first segment of Γ_n in Figure 4 (once the choice is made, the sides alternate).

The first statement of Theorem 2 also holds for all generalized Dragon curves: they never cross themselves. Figure 7 shows some specimen from the zoo of generalized Dragons.

The topic of Dragon curves is wide and deep. Here are some pointers to the literature. The reader interested in paper folding sequences as automated sequences is referred to the book [1]. The relation of Dragon curves with the Rudin-Shapiro sequences is discussed in [5]–[7] and [12]. A connection to the binary Gray code is made in [2]. For a version of 3-dimensional paper folding (wire bending), see [13]. Many new kinds of Dragon-like curves, their self-avoiding and plane filling properties are described in [8]; this recent paper is based on the results obtained by its author in 1975.

Open problems abound, and we finish by mentioning one. The following is another quotation from the addendum in [10]:

> While preparing the figure which opens up the dragon-sequence folds to angles of $100°$ at each bend, I noticed in 1969 that $95°$-

[3] The reader interested in the relation of Dragon curves with statistical mechanics, in particular, the Ising model, is referred to [11, 12].

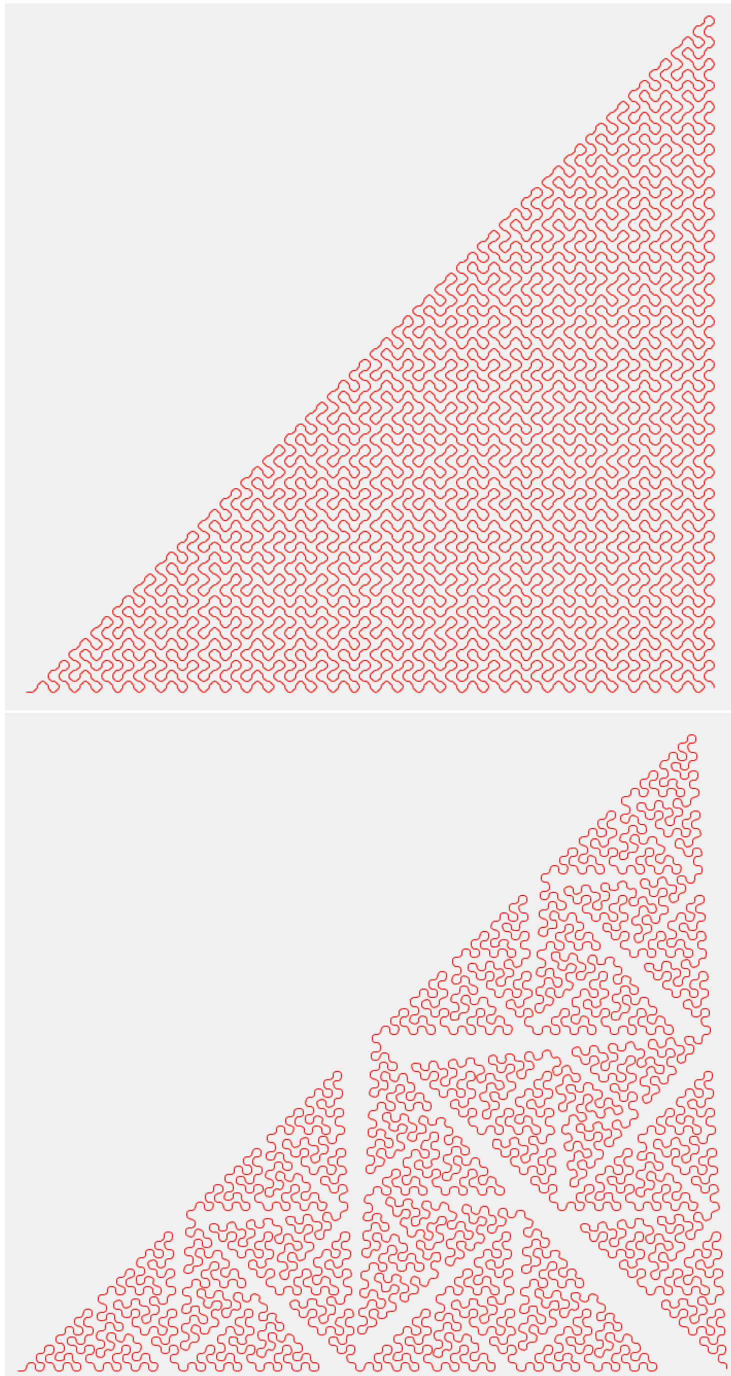

Figure 6: Dragon curves corresponding to the alternating folding: opening angles $\pi/2$ and $33\pi/64$.

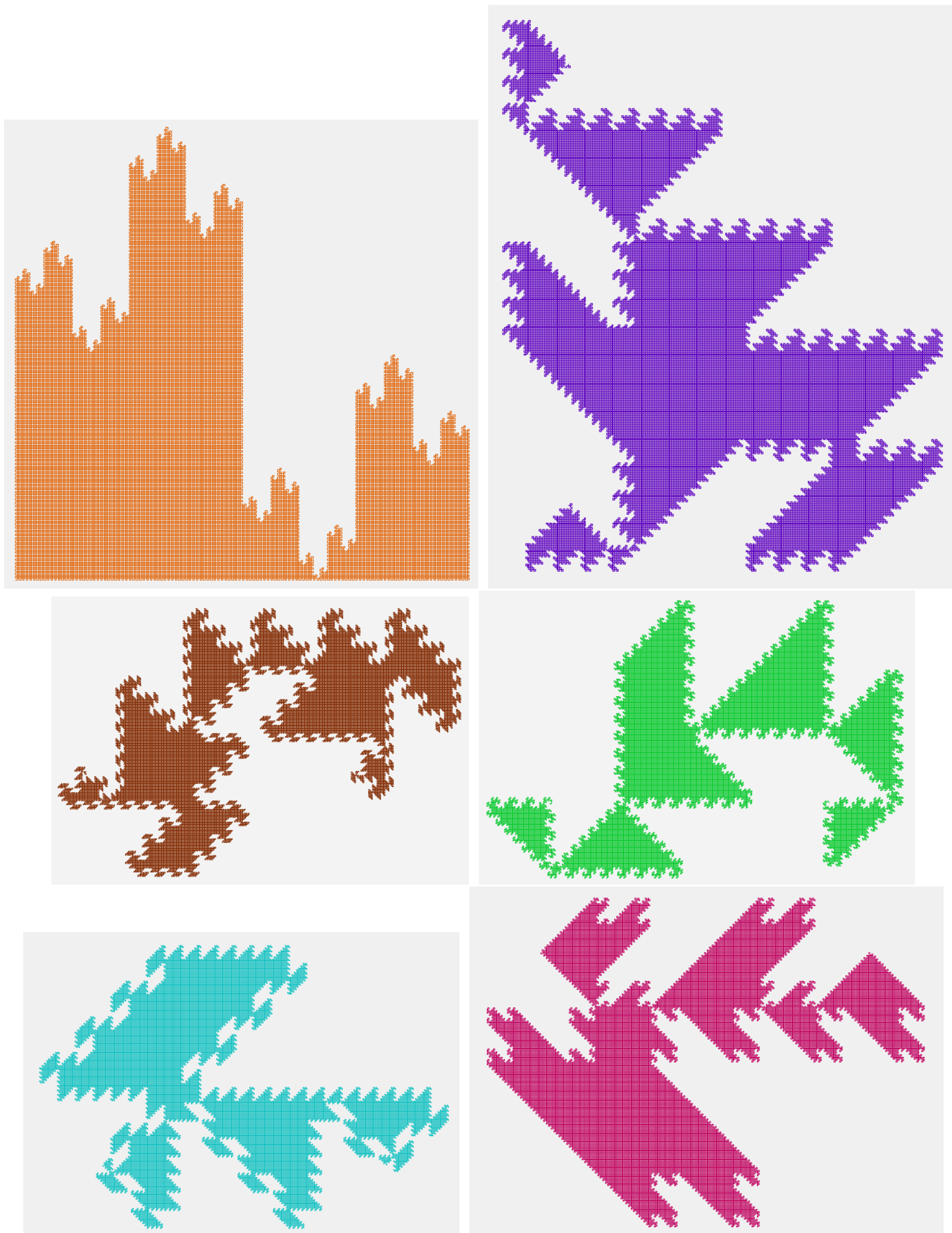

Figure 7: Various generalized Dragon curves.

angle folds would lead to paths that cross themselves. For example, the path obtained from S_{10} will interfere with itself just before points 447 and 703; and if we look further, 95° bends applied to S_{12} will yield a party that crosses itself quite dramatically before and after points 1787 and 2807.

This phenomenon, illustrated in Figure 8, needs an explanation. In particular, what is the value of the critical angle for which the curve starts to cross itself and where does this self-crossing occur?

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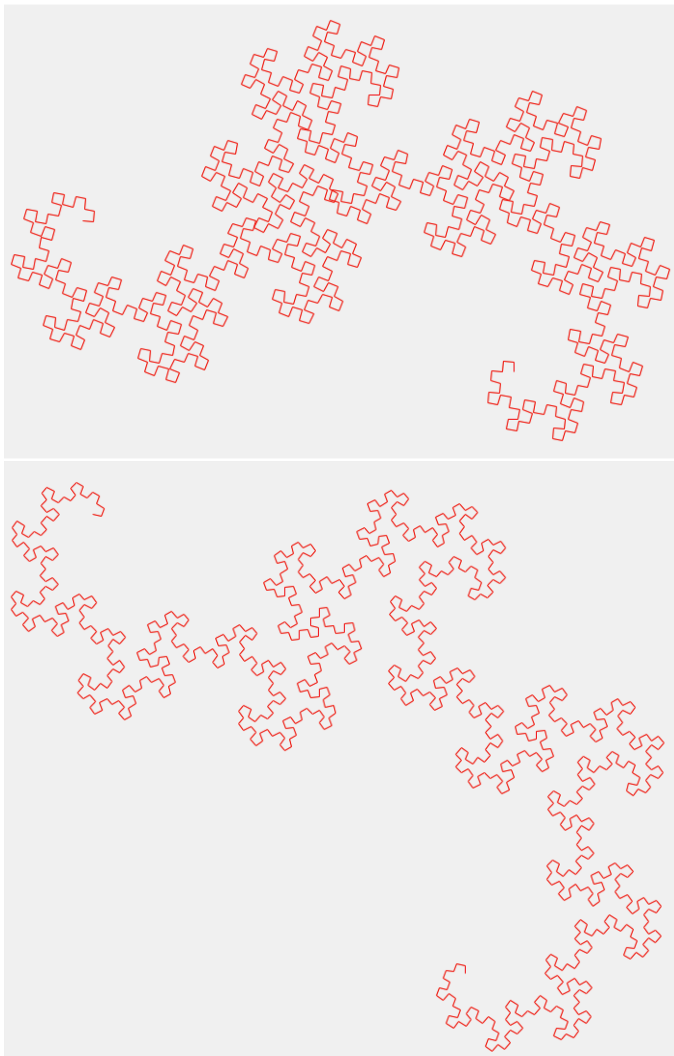

Figure 8: Self-intersection – in the middle – of the 10th generation Dragon curve with the opening angle of about 94°. The non self-intersecting curve has the opening angle of about 100°.

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