The Six Circles Theorem revisited

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Introduction. Given a triangle $P_1P_2P_3$, construct a chain of circles: C_1, inscribed in the angle P_1; C_2, inscribed in the angle P_2 and tangent to C_1; C_3, inscribed in the angle P_3 and tangent to C_2; C_4, inscribed in the angle P_1 and tangent to C_3, and so on. The claim of The Six Circles Theorem is that this process is 6-periodic: $C_7 = C_1$, see Figure 1.

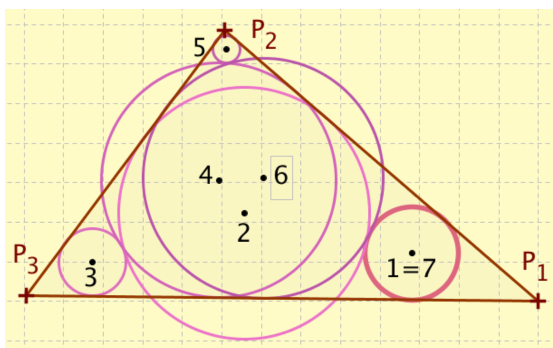

Figure 1: The Six Circles Theorem: the centers of the consecutive circles are labeled $1, 2, \ldots, 7$.

This beautiful theorem is one of many in the book [3] which is a result of collaboration of three geometry enthusiasts, C. Evelyn, G. Money-Coutts, and J. Tyrrell. The following is a quotation from John Tyrrell's obituary [6]:

> John also worked with two amateur mathematicians, C. J. A. Evelyn and G. B. Money-Coutts, who found theorems by using outsize drawing instruments to draw large figures. They then

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> looked for concurrencies, collinearities, or other special features. The three men used to meet for tea at the Cafe Royal and talk about mathematics, and then go to the opera at Covent Garden, where Money-Coutts had a box.

We refer to [12, 8, 4, 9, 10] for various proofs and generalizations and to [11] for a brief biography of C. J. A. Evelyn. See also [2, 13, 14] for Internet resources.

A refinement. The formulation of the Six Circles Theorem needs clarification. Firstly, there are two choices for each next circle; we assume that each time the smaller of the two circles tangent to the previous one is chosen (that is, the one which is closer to the respective vertex of the triangle). Secondly it well may happen that the next circle is tangent not to a side of the triangle but rather to its extension.

The Six Circles Theorem, as stated at the beginning, holds for a chain of circles for which all tangency points lie on the sides of the triangle, not their extensions. And what about the latter case? Figure 2 shows what may happen.

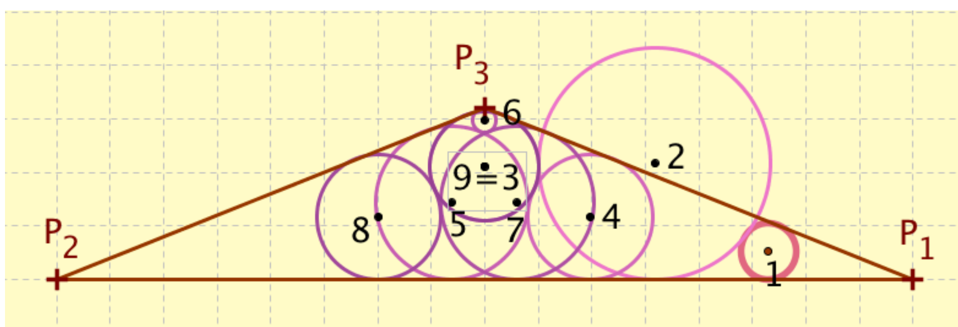

Figure 2: The chain of circles is eventually 6-periodic with pre-period of length two: $C_9 = C_3$, but $C_8 \neq C_2$.

Theorem 1 *Assume that, for the initial circle, at least one of the tangency points lies on a side of the triangle. Then the chain of circles is eventually 6-periodic. One can choose the shape of a triangle and an initial circle so that the pre-period is arbitrarily long.*

The existence of pre-periods is due to the fact that the map assigning the next circle to the previous one is not 1-1, that is, the inverse map is multi-valued.

Concerning the assumption that at least one of the tangency points of a circle with the sides of the angle of a tirangle lies on a side of the triangle, and not its extension, we observe the following.

Lemma 2 *If the first circle in the chain satisfies this assumption then so do all the consecutive circles.*

Proof. If circle C_1 touches side P_1P_2 then circle C_2 also touches this side, at a point closer to P_2 than the previous tangency point. Shifting the index by one, if circle C_2 does not touch side P_2P_3 but touches side P_1P_2 then it intersects side P_1P_3, and the next circle C_3 touches side P_1P_3, at a point closer to P_3 than the intersection points. See Figure 5 below for an illustration. □

What about the case when the initial circle touches the extensions of both sides, P_1P_2 and P_1P_3? If the circle does not intersect side P_2P_3 then the next circle in the chain cannot be constructed, so this case is not relevant to us. If the first circle intersects side P_2P_3 then the next circle touches side P_2P_3, and thus satisfies the assumption of Theorem 1, see Figure 3. Hence this assumption holds, starting with the second circle in the chain, and we may make it without loss of generality.

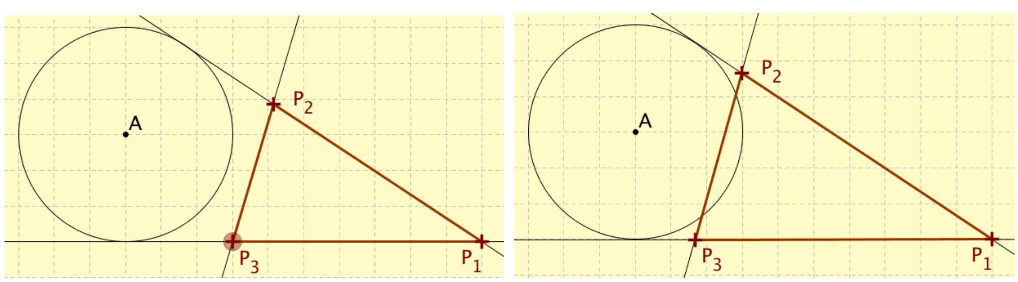

Figure 3: When the initial circle touches the extensions of both sides of the triangle.

Beginning of the proof. The proof consists of reducing the system to iteration of a piecewise linear function; this is achieved by a trigonometric change of variables (see [3, 9, 10, 4] for versions of this approach). The choices of coordinates and various manipulations may look somewhat unmotivated; they are merely justified by the fact that they work. The reader interested in a coordinate-free, but less elementary, approach is referred to [9].

Let us introduce notations. The angles of the triangle are $2\alpha_1, 2\alpha_2$ and $2\alpha_3$; its side lengths are a_1, a_2, a_3 (with the usual convention that ith side is opposite ith vertex). Let $p = (a_1 + a_2 + a_3)/2$. We note that $p > a_i$ for $i = 1, 2, 3$: this is the triangle inequality. We denote the radii of the circles C_i by r_i, $i = 1, 2, \ldots$ and assume that C_i is a circle that is inscribed into (i mod 3)-rd angle.

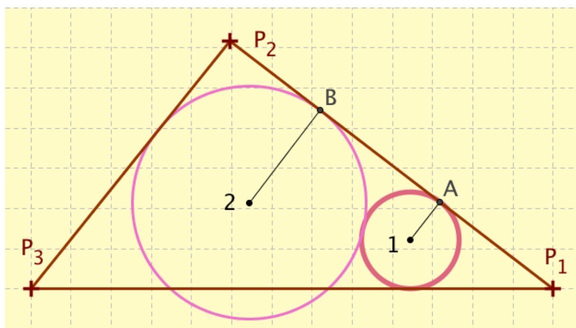

Figure 4: The first case of equation (1): $|P_1A| + |AB| + |BP_2| = |P_1P_2|$.

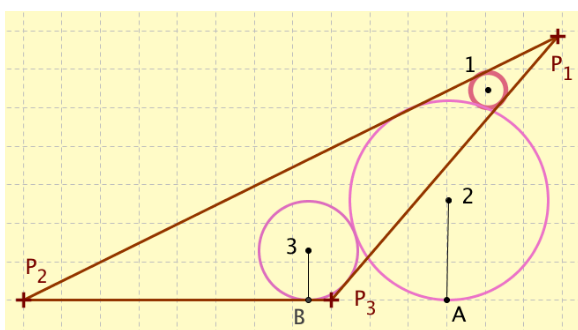

Figure 5: The second case of equation (1): $|P_2A| - |AB| + |BP_3| = |P_2P_3|$.

If two circles of radii r_1 and r_2 are tangent externally then the length of their common tangent segment (segment AB in Figures 4, 5, 6) is

$$\sqrt{(r_1 + r_2)^2 - (r_1 - r_2)^2} = 2\sqrt{r_1 r_2}.$$

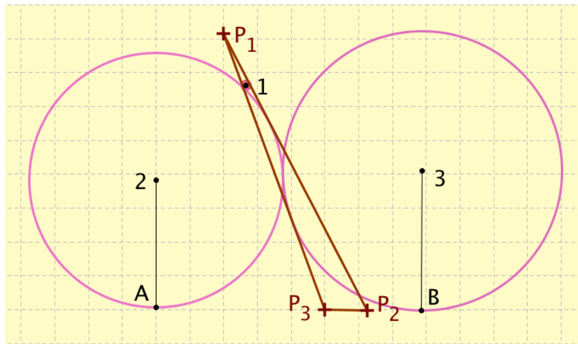

Figure 6: Also the second case of equation (1): $|P_2A|-|AB|+|BP_3| = |P_2P_3|$.

Thus, depending on the mutual positions of the consecutive circles, as shown in Figures 4, 5 and 6, we obtain the equations

$$r_1 \cot\alpha_1 + 2\sqrt{r_1r_2} + r_2 \cot\alpha_2 = a_3 \;\text{ or }\; r_1 \cot\alpha_1 - 2\sqrt{r_1r_2} + r_2 \cot\alpha_2 = a_3, \quad (1)$$

or the cyclic permutation of the indices $1, 2, 3$ thereof. Specifically, if C_1 is tangent to the side P_1P_2 then we have the first equation (1), and if C_1 is tangent to the extension side P_1P_2 then we have the second equation.

Solving the equations. Equations (1) determine the new radius r_2 as a function of the previous one, r_1. We shall solve these equations in two steps. First, introduce the notations

$$u_1 = \sqrt{r_1 \cot\alpha_1}, \quad e_3 = \sqrt{\tan\alpha_1 \tan\alpha_2},$$

and their cyclic permutations. Then (1) is rewritten as

$$u_1^2 \pm 2e_3u_1u_2 + u_2^2 = a_3, \quad (2)$$

or

$$u_1(u_1 \pm e_3u_2) + u_2(u_2 \pm e_3u_1) = a_3. \quad (3)$$

Solve (2) for u_2:

$$u_2 = -e_3u_1 + \sqrt{a_3 - (1-e_3^2)u_1^2}, \;\text{ or }\; u_2 = e_3u_1 - \sqrt{a_3 - (1-e_3^2)u_1^2}, \quad (4)$$

according as the sign in (2) is positive or negative. The minus sign in front of the radical in the second formula (4) is because our construction chooses the smaller of the two circles tangent to the previous one. Likewise, solve for u_1:

$$u_1 = -e_3 u_2 + \sqrt{a_3 - (1 - e_3^2) u_2^2}, \quad \text{or} \quad u_1 = e_3 u_2 + \sqrt{a_3 - (1 - e_3^2) u_2^2},$$

again depending on the sign in (2). The plus sign in front of the radical in the second formula is due to the fact that, going in the reverse direction, from C_2 to C_1, one chooses the greater of the two circles. Substitute to (3) to obtain

$$u_1 \sqrt{a_3 - (1 - e_3^2) u_2^2} \pm u_2 \sqrt{a_3 - (1 - e_3^2) u_1^2} = a_3. \tag{5}$$

The sign depends on whether u_1^2 is smaller or greater than a_3 (and if $u_1^2 = a_3$ then $u_2 = 0$ in (4)).

Trigonometric substitution. We shall rewrite the previous formula as the formula for sine of the sum or difference of two angles. To do so, we need a lemma.

Given a triangle ABC, let a, b, c be its sides, p its semi-perimeter, and $2\alpha, 2\beta, 2\gamma$ its angles.

Lemma 3 *One has*

$$1 - \tan\alpha \tan\beta = \frac{c}{p}.$$

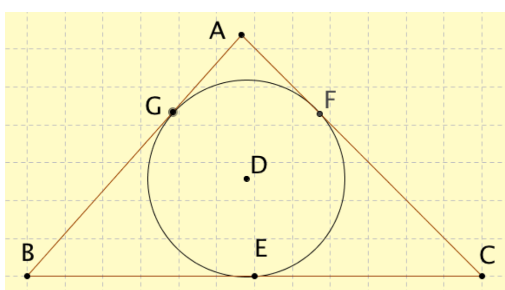

Figure 7: To proof of Lemma 3.

Proof. Let R be the inradius and S the area of the triangle. Let

$$T_A = AF = AG,\ T_B = BG = BE,\ T_C = CE = CF,$$

see Figure 7. Then $p = T_A + T_B + T_C$, and $S = Rp$. By Heron's formula, $S = \sqrt{pT_AT_BT_C}$. Therefore $R^2p = T_AT_BT_C$.

On the other hand, $\tan\alpha = R/T_A, \tan\beta = R/T_B$, hence

$$1 - \tan\alpha\tan\beta = 1 - \frac{R^2}{T_AT_B} = 1 - \frac{T_C}{p} = \frac{T_A + T_B}{p} = \frac{c}{p},$$

as claimed. □

Using the lemma, we rewrite (5) as

$$\frac{u_1}{\sqrt{p}}\sqrt{1 - \frac{u_2^2}{p}} \pm \frac{u_2}{\sqrt{p}}\sqrt{1 - \frac{u_1^2}{p}} = \sqrt{\frac{a_3}{p}}. \tag{6}$$

We are ready for the final change of variables. Let

$$\varphi_i = \arcsin\left(\frac{u_i}{\sqrt{p}}\right),\ \beta_i = \arcsin\left(\sqrt{\frac{a_i}{p}}\right).$$

To justify the second formula, we note that $a_i < p$. Likewise, each circle is tangent to a side of the triangle, so u_i^2 is not greater than some side, and hence less than p. This justifies the first formula.

In the new variables, (6) rewrites as $\sin(\varphi_1 \pm \varphi_2) = \sin\beta_3$, where one has plus sign for $\varphi_1 < \beta_3$ and minus sign otherwise. Hence

$$\varphi_2 = |\varphi_1 - \beta_3|. \tag{7}$$

This equation describes the dynamics of the chain of circles.

Before studying the dynamics of this function we note that the angles β_i satisfy the triangle inequality, as the next lemma asserts. Assume that $\beta_1 \le \beta_2 \le \beta_3$.

Lemma 4 *One has* $\beta_3 < \beta_1 + \beta_2$.

Proof. We start by noting that $\sin \beta_i < 1$ for $i = 1, 2, 3$, and that

$$\sin^2 \beta_1 + \sin^2 \beta_2 + \sin^2 \beta_3 = 2 \quad \text{or} \quad \sin^2 \beta_3 = \cos^2 \beta_1 + \cos^2 \beta_2.$$

Assume that the triangle inequality is violated for some triangle. Since the inequality holds for an equilateral triangle, one can deform it to obtain a triangle for which $\beta_3 = \beta_1 + \beta_2$. Then

$$\cos^2 \beta_1 + \cos^2 \beta_2 = \sin^2 \beta_3 = (\sin \beta_1 \cos \beta_2 + \sin \beta_2 \cos \beta_1)^2.$$

It follows, after some manipulations, that

$$\sin \beta_1 \sin \beta_2 \cos \beta_1 \cos \beta_2 = \cos^2 \beta_1 \cos^2 \beta_2 \quad \text{or} \quad \sin \beta_1 \sin \beta_2 = \cos \beta_1 \cos \beta_2.$$

Therefore

$$\cos(\beta_1 + \beta_2) = 0 \quad \text{or} \quad \beta_1 + \beta_2 = \frac{\pi}{2}.$$

Hence $\sin^2 \beta_1 + \sin^2 \beta_2 = 1$, and thus $\sin \beta_3 = 1$. This is a contradiction. □

Piecewise linear dynamics. We are ready to investigate the function (7). Although the dynamics of a piecewise linear function can be very complex [7], ours is quite simple.

Iterating the map three times, with the values of the index $i = 1, 2, 3$, yields the function $y = |||x - \beta_1| - \beta_2| - \beta_3|$. We scale the xy plane so that $\beta_1 = 1$ and rewrite the function as

$$f(x) = |||x - 1| - a| - b| \tag{8}$$

where $a \le b$ and $b < a + 1$. We will show that every orbit of the map f is eventually 2-periodic, see Figure 8.

The graph of $f(x)$ is shown in Figure 9 with the characteristic points marked.

It is clear that iterations of the function f take every orbit to the segment $[0, b]$, and this segment is mapped to itself. Indeed, if $x \ge a + b + 1$ then $f(x) = x - a - b - 1$, and if $x \le a + b + 1$ then $f(x) \le b$. Thus iterations of the function f will keep decreasing x until it lands on $[0, b]$.

Let

$$I_1 = [0, b - a], \; I_2 = [b - a, 1], \; I_3 = [1, b].$$

Then I_2 consists of 2-periodic points, and we need to show that every orbit lands on this interval. Indeed, $f(I_1) = [1, b - a + 1] \subset I_3$. On the other hand,

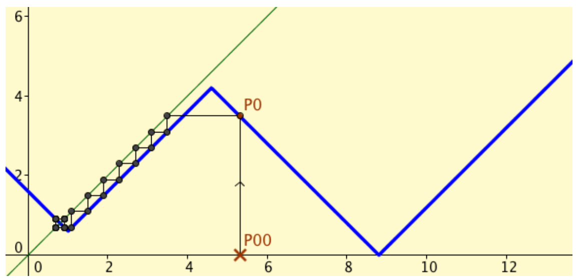

Figure 8: Iteration of function $f(x)$ for $a = 3.6, b = 4.2$.

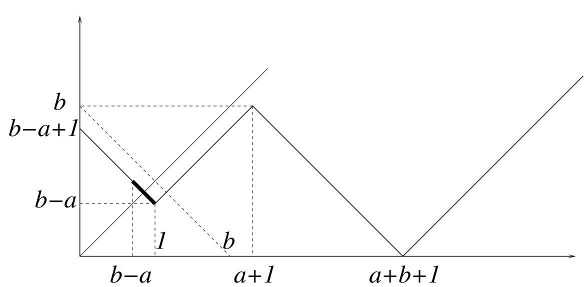

Figure 9: The graph $y = f(x)$. The segment $[b - a, 1]$ consists of 2-periodic points.

each iteration of f "chops off" from the left a segment of length $1 + a - b$ from I_3 and sends it to I_2. It follows that every orbit eventually reaches I_2.

If $|I_2| = a + 1 - b$ is small, it may take an orbit a long time to reach I_2. For example, take $a = 1$ and $b = 2 - \varepsilon$. Then, choosing ε sufficiently small, one can make the pre-period of point $x = \varepsilon$ arbitrarily long. This choice corresponds to an isosceles triangle with the obtuse angle close to π and a small initial circle C_1, compare with Figure 2.

Final comments.

1) Although our considerations are close to those in [3], the authors of this book did not consider the pre-periodic behavior of the chain of circles. They addressed the issue of the two choices in each step of the construction and noted:

> ... we may make the first three sign choice quite arbitrarily provided that, thereafter, we make 'correct' choices ...

so that the chain becomes 6-periodic.

2) For a parallelogram, a similar phenomenon holds: the chain of circles is eventually 4-periodic but with a pre-period, see [10]. Our analysis is similar to that of Troubetzkoy.

3) For $n > 3$, the chain of circles inscribed in an n-gon is generically chaotic, see [10] for a proof when $n = 4$ and Figure 10 for an illustration when $n = 5$. However, for every n, there is a class of n-gons enjoying $2n$-periodicity, see [9]. Presumably, this periodicity is also eventual, with an arbitrary long pre-period.

4) A version of the Six Circles Theorem holds for curvilinear triangles made of arcs of circles [3, 12, 8], and a generalization to n-gons is available as well [9]. Again, one expects eventual periodicity with arbitrarily long pre-periods.

5) Constructing the chains of circles, we consistently chose the smaller of the two circles tangent to the previous one. It is interesting to investigate what happens when other choices are made; for example, one may toss a coin at each step. See Figure 11 for an experiment with a randomly chosen triangle.

6) The Six Circles Theorem is closely related with the Malfatti Problem: to inscribe three pairwise tangent circles into the three angles of a triangle; see, e.g., [5] and the references therein. This 3-periodic chain of circles exists and is unique for every triangle; it corresponds to the fixed point of the

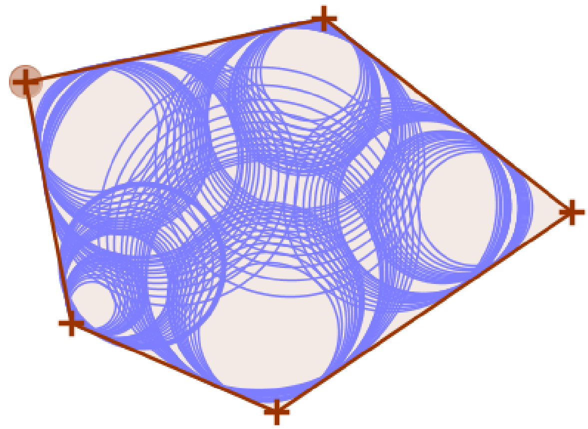

Figure 10: A chain of circles in a pentagon.

function $f(x)$. See [1] for a discussion of the Malfatti Problem close to our considerations.

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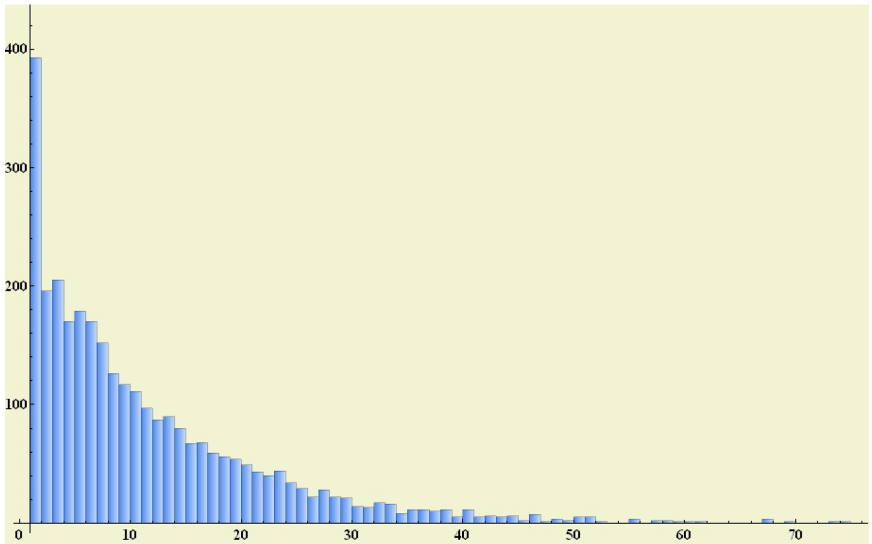

Figure 11: The histogram represents 3000 chains of circles in a generic triangle. The selection, out of two, of each next circle in a chain is random. The horizontal axis represents the length of the pre-period, and the vertical the number of chains having this pre-period.

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