INTEGRALS STRIKE BACK

Directions:
- To do each of these indefinite integrals you need in your answer no more than
  - The original radical(s) in the integrand.
  - Extra constants like $\sqrt{2}$ or $\sqrt{3}$.
  - The natural log and inverse tangent functions defined by
    \[
    \ln(x) = \int_1^x \frac{dt}{t},
    \]
    \[
    \tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2},
    \]
    respectively.
  - Other inverse functions such as $\sin^{-1}(x)$ may be convenient but are not strictly necessary.
- The integrals are arranged in order of increasing difficulty. You may get the first one rather quickly but
  easily run into difficulties down the list. The last three are very difficult. ¹
- Hints are provided on the third page and answers are provided on the back of this page. If you absolutely
  cannot solve one of these integrals, you can at least verify the correctness of the answer by differentiating.

1
\[
\int \sqrt{1-x^2} \, dx
\]

(2)
\[
\int \frac{dx}{x + \sqrt{1+x^2}}
\]

(3)
\[
\int \frac{\sqrt{x}}{\sqrt{1-x^3}} \, dx
\]

(4)
\[
\int \frac{6(x^3+1)^{5/2}}{x^3} \, dx
\]

(5)
\[
\int \frac{dx}{x^{1/4}(1-x)^{3/4}}
\]

(6)
\[
\int (1-x^3)^{2/3} \, dx
\]

(7)
\[
\int \frac{dx}{\sqrt{x}\sqrt{x + \sqrt{x + \sqrt{x}}}}
\]

(8)
\[
\int \frac{2x}{\sqrt{x^4 + 4x^2 + 5}} \, dx
\]

(9)
\[
\int \frac{3x - 5}{\sqrt{x(x-4)(x-3)(x+5)}} \, dx
\]

(10)
\[
\int \frac{4x - 2}{\sqrt{x^2 - 6x + 4\sqrt{x^2 + 2x + 4}}} \, dx
\]

(11)²
\[
\int \frac{6x}{\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1}} \, dx
\]

¹ At the time of writing, www.wolframalpha.com is able to do six out of the eleven integrals, for a score of 55%.
² This source of this integral is unknown; it appeared on the main chalkboard of McAllister in the Spring of 2016 and motivated the three previous simpler integrals.
ANSWERS

Many answers may be correct. Here are some fully simplified suggestions without the +C. The answer to (5) is

most elegantly stated in terms of the inverse hyperbolic tangent, defined by

\[
\tanh^{-1}(x) = \int_0^x \frac{dt}{1-t^2} = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right).
\]

(1)

\[
\int \sqrt{1-x^2} dx = \frac{1}{2} x \sqrt{1-x^2} + \tan^{-1} \left( \frac{1+x}{\sqrt{1-x^2}} \right)
= \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x)
\]

(2)

\[
\int \frac{dx}{x + \sqrt{1+x^2}} = -\frac{1}{2} x^2 + \frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \ln |x + \sqrt{1+x^2}|
\]

(3)

\[
\int \frac{\sqrt{x}}{\sqrt{1-x^2}} dx = \frac{2}{3} \tan^{-1} \left( \frac{x^{3/2}}{\sqrt{1-x^2}} \right) = \frac{2}{3} \sin^{-1} \left( x^{3/2} \right)
\]

(4)

\[
\int \frac{6(x^2+1)^{5/2}}{x^3} dx = \frac{(2x^4+14x^2-3)\sqrt{x^2+1}}{x^2} + 15 \ln \left| \frac{x}{\sqrt{x^2+1}+1} \right|
\]

(5)

\[
\int \frac{dx}{x^{1/4}(1-x)^{3/4}} = \sqrt{2} \tan^{-1} \left( \frac{\sqrt{2}x^{1/4}(1-x)^{1/4}}{\sqrt{1-x^2}\sqrt{x}} \right) - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{1-x+\sqrt{x}+\sqrt[4]{x}}}{\sqrt{1-x+\sqrt{x}-\sqrt[4]{x}}(1-x)^{1/4}} \right|
\]

\[
= \sqrt{2} \tan^{-1} \left( \frac{\sqrt{2}x^{1/4}(1-x)^{1/4}}{\sqrt{1-x^2}\sqrt{x}} \right) - \sqrt{2} \tan^{-1} \left( \frac{\sqrt{2}x^{1/4}(1-x)^{1/4}}{\sqrt{1-x^2}\sqrt{x}} \right)
\]

(6)

\[
\int (1-x^3)^{2/3} dx = \frac{1}{3} x(1-x^3)^{2/3} + \frac{1}{3} \ln \left| x + (1-x^3)^{1/3} + \frac{2}{3} \sqrt[3]{x} \tan^{-1} \left( \frac{x-2(1-x^3)^{1/3}}{\sqrt[3]{3x}} \right) \right|
\]

(7)

\[
\int \frac{dx}{\sqrt{x} \sqrt{\sqrt{x} + \sqrt{x} + \sqrt{x}}} = \left( 1-2\sqrt{x} + 2\sqrt{x+\sqrt{x}} \right) \sqrt{\sqrt{x} + \sqrt{x+\sqrt{x}}}
- \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{\sqrt{2}\sqrt{x} + \sqrt{x+\sqrt{x}}}{\sqrt{x} + \sqrt{x+\sqrt{x}}} \right)
\]

(8)

\[
\int \frac{2x}{\sqrt{x^2 + 4x^2 + 5}} dx = \ln \left| x^2 + 2 + \sqrt{x^4 + 4x^2 + 5} \right|
\]

(9)

\[
\int \frac{3x-5}{\sqrt{x(x-4)(x-3)(x+5)}} dx = \ln \left| x^3 + 2x^2 - 15x - 18 + (x+3)\sqrt{x(x-4)(x-3)(x+5)} \right|
\]

(10)

\[
\int \frac{4x-2}{\sqrt{x^2 - 6x + 4\sqrt{x^2 + 2x + 4}}} dx = \ln \left| x^4 - 6x^3 + 4x^2 + 32 + x(x-4)\sqrt{x^2 - 6x + 4\sqrt{x^2 + 2x + 4}} \right|
\]

(11)

\[
\int \frac{6x}{\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1}} dx = \ln \left| x^6 + 12x^5 + 45x^4 + 44x^3 - 33x^2 + 43 \right. + \left( x^4 + 10x^3 + 30x^2 + 22x - 11 \right)\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1} \right|
\]
HINTS

There are many possible avenues of attack. Here are some of the most accessible options. Note that in order to use these hints, you will have to

- accurately perform the algebraic manipulations required to apply the substitution
- integrate the resulting trigonometric integral in $\theta$ or rational/algebraic function in $u$ using the standard techniques.

1. Substitute $x = \sin(\theta)$.

2. Rationalize the denominator, then substitute $x = \tan(\theta)$.

3. Substitute $x^3 = \sin^2(\theta)$.

4. Substitute $x = \tan(\theta)$ then $u = \cos(\theta)$.

5. Rewrite the integrand slightly and use $u = \frac{(1-x^3)^{-1}}{x}$. Note that $u^4 + 1$ can be factored into two quadratics if you use $\sqrt{2}$.

$$u^4 + 1 = (u^2 + 1)^2 - (\sqrt{2} u)^2.$$ 

6. This is similar to the previous problem. The substitution $u = \frac{(1-x^3)^{-1}}{x}$ works well. Note that you may first want to use integration by parts and some clever manipulations to convert to the integral to

$$\int \frac{dx}{(1-x^3)^{1/3}},$$

where the same substitution works.

7. Substitute $u = \sqrt[3]{x + \sqrt{x} + \sqrt[3]{x}}$. It is actually feasible to solve for $x$ here.

8. Substitute $u = x^2$ and the resulting integral falls to the standard techniques.

9. Multiply top and bottom by $(x + 3)$ and move the $(x + 3)$ inside the square root. Now substitute $u = x(x - 3)(x - 5)$.

10. Try a similar technique to previous problem. Multiply top and bottom by $x(x - 4)$. The substitution for $u$ should be fourth degree in $x$.

11. This problem can done by first applying a usual trigonometric substitution and then applying the trick from the previous two problems. Note that $\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1} = \sqrt{(x + 1)^2 - 12x^2}$, so try the substitutions $x \to \theta \to u$ defined by

$$\frac{(x + 1)^2}{\sqrt{12x}} = \sec \theta = \frac{u}{\sqrt{3}}\

As the solution for $x$ is $x = \sqrt{u^2 - 2} + u - 1$, the integral becomes

$$\int \frac{6x}{\sqrt{x^4 + 4x^3 - 6x^2 + 4x + 1}} dx = \int 3 \sec \theta + \sqrt{3} \frac{\sqrt{3} \sec \theta - 1}{\sqrt{3} \sec \theta - 2} d\theta$$

$$= \int 3 \sec \theta d\theta + \int \frac{3(u - 1)du}{\sqrt{u(u - 2)}\sqrt{u^2 - 3}}$$

$$= \int 3 \sec \theta d\theta + \int \frac{3u^2 - 3}{(u + 1)^2(u - 2)u(u^2 - 3)} du$$

$$= \int 3 \sec \theta d\theta + \int \frac{3u^2 - 3}{u^3 - 3u - 2}(u^3 - 3u - 1) du$$

$$= \int 3 \sec \theta d\theta + \int \sec \phi d\phi \quad \text{set} \quad \sec \phi = u^3 - 3u - 1$$