DIMENSION OF NON-CONFORMAL REPELLERS: A SURVEY

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Abstract. This article is a survey of recent results on dimension of repellers for expanding maps and limit sets for iterated function systems. While the case of conformal repellers is well understood the study of non-conformal repellers is in its early stages though a number of interesting phenomena have been discovered, some remarkable results obtained and several interesting examples constructed. We will describe contemporary state of the art in the area with emphasis on some new emerging ideas and open problems.

Contents

1. Introduction

The dimension of invariant sets is among the most important characteristics of dynamical systems. In this paper, we describe some recent advances in

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dimension theory of repellers for expanding maps focusing mostly on non-conformal repellers. We emphasize that the case of conformal repellers for which at each point the rates of expansion are the same in all directions in the space, is essentially complete: various dimension characteristics of the repeller (e.g., the Hausdorff dimension, lower and upper box dimensions) all agree and the common value is given as a root of famous Bowen's equation. In addition, there exists a unique measure of maximal dimension and the Hausdorff measure at dimension is finite.

The study of dimension of non-conformal repellers has proven to be much more difficult due to distinct rates of expansion in different directions in the space leading to distinct values of the Lyapunov exponent. Serious obstacles include complicated geometric shape of the images of small balls under the iterations of the expanding map (see [11, 15, 23]) as well certain number-theoretical properties of expansion coefficients, which cause the variation of the Hausdorff dimension with respect to a certain generic value (see [32]).

So far a substantial progress has been made in studying dimension of non-conformal repellers of some particular types especially the ones generated by *Iterated Function Systems* (IFS). This includes self-affine sets, which can be viewed as repellers of some expanding maps provided the open set condition of Hutchinson holds. This study emerged from the classical work of Bedford [5] and of McMullen [26] on *generalized Sierpiński carpets*. Already in these examples one can observe such a nontrivial phenomenon as non-coincidence of Hausdorff and box dimensions. These "pathological" examples prompted attempts to show that "generically" these dimensions should coincide. To this end Falconer, [9] derived an implicit formula for the Hausdorff and box dimensions of "almost all" self-affine sets (though under certain technical restrictions). Some attempts have been made to extend results by Falconer to limit sets associated with iterated function systems generated by non-linear contractions, [22, 23].

Since finding an explicit formula for the Hausdorff dimension of non-conformal repellers (like Bowen's formula in the conformal case) seems to be a daunting task, one can try to obtain lower and upper bounds for the dimension. Indeed, sharp dimension estimates on dimension may hint an appropriate explicit formula. Those bounds are also important in the study of some infinite-dimensional dynamical systems.

The structure of the paper is as follows. In Section 2 we briefly recall some necessary notions and results from the dimension theory of conformal repellers. In Section 3, we present various examples of non-conformal repellers, appearing as results of Iterated Function Systems including generalized Sierpiński carpets, generalized Sierpiński sponges and some self-affine sets, and we describe several results on computing their Hausdorff and box dimensions. In Section 4, we discuss dimension estimates for some non-conformal repellers.

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2. Conformal Repellers

Let $f : M \to M$ be a $C^{1+\alpha}$ map of a smooth Riemannian manifold and $\Lambda \subset M$ a compact invariant subset, i.e., $f^{-1}(\Lambda) = \Lambda$. We say that f is *expanding* on Λ and Λ is a *repeller* for f if

(1) there exists an open neighborhood V of Λ (called a *basin*) such that
$$\Lambda = \{x \in V : f^n(x) \in V \text{ for all } n \geq 0\};$$
(2) there exist $C > 0$ and $\lambda > 1$ such that $\|d_x f^n v\| \geq C\lambda^n \|v\|$ for all $x \in \Lambda$, $v \in T_xM$ and $n \in \mathbb{N}$.

A smooth map $f : M \to M$ is called *conformal* if $d_x f = a(x)\mathrm{Isom}_x$ for each $x \in M$, where Isom_x is an isometry of T_xM and $a(x)$ is a Hölder continuous function on M. A conformal map f is *expanding* if $|a(x)| > 1$ for all $x \in M$. A repeller Λ for a conformal expanding map is called a *conformal repeller.*

There are many interesting examples of conformal repellers, such as *hyperbolic Julia sets* and *dynamically defined Cantor sets*; see [30] for more details.

Recall that the *s-dimensional Hausdorff measure* of a set $F \subset \mathbb{R}^d$ is defined by
$$\mathcal{H}^s(F) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{j=1}^{\infty} |U_j|^s : \ F \subset \bigcup_{j=1}^{\infty} U_j, \ |U_j| \leq \varepsilon \right\},$$
where $|\cdot|$ denotes the diameter of a set. Then
$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$
is called the *Hausdorff dimension* of F. Furthermore, if μ is a Borel finite measure on $\mathbb{R}^d$, its *Hausdorff dimension* is defined to be
$$\dim_H \mu = \inf\{\dim_H Z : Z \subset \mathbb{R}^d \text{ with } \mu(Z) = 1\}.$$
A measure μ supported on a set F is called a measure of *full Hausdorff dimension* if $\dim_H \mu = \dim_H F$.

Let $N_\varepsilon(F)$ be the smallest number of sets of diameter ε needed to cover a non-empty bounded set F. Then the *lower* and *upper box dimensions* of F are
$$\underline{\dim}_B F = \underline{\lim}_{\varepsilon \to 0} \frac{\log N_\varepsilon(F)}{-\log \varepsilon}, \text{ and } \overline{\dim}_B F = \overline{\lim}_{\varepsilon \to 0} \frac{\log N_\varepsilon(F)}{-\log \varepsilon}.$$

One can see that $\dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F$ for any non-empty bounded set F. For more information on dimensions, see [8], [30].

We need some definitions from thermodynamic formalism. Let (X, d) be a compact metric space, $f : X \to X$ a continuous transformation and $\phi : X \to \mathbb{R}$ a continuous function. Write $\phi_n = \sum_{k=0}^{n-1} \phi \circ f^k$. We call $E \subset X$ an (n, ε)-*separating* subset if for any $x, y \in E$, $x \neq y$, we have

$$d_n(x, y) := \max_{0 \le k \le n-1} d(f^k(x), f^k(y)) > \varepsilon.$$

Then we define the *topological pressure* for (X, f, ϕ) as follows

$$P_X(f, \phi) := \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp \phi_n(x) : \ E \text{ is } (n, \varepsilon)\text{-separating} \right\}.$$

See [37] for other equivalent definitions and [30] for "dimensional" definitions of topological pressure in more general situations.

Given an f-invariant subset $Z \subset X$, let $\mathcal{M}(Z)$ be the set of all f-invariant Borel probability measures supported on Z. We call $m \in \mathcal{M}(Z)$ an *equilibrium measure* on Z corresponding to the function ϕ if

$$h_m(f) + \int_Z \phi \, dm = \sup_{\mu \in \mathcal{M}(Z)} \left(h_\mu(f) + \int_Z \phi \, d\mu \right),$$

where $h_\mu(f)$ is the measure-theoretic entropy.

We now present some dimension results on conformal repellers.

Theorem 2.1. *Let $f : M \to M$ be a $C^{1+\alpha}$ conformal expanding map with a conformal repeller Λ and $d_x f = a(x) Isom_x$ for any $x \in M$. Then*

(1) $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = s$, *where s is the unique root of Bowen's equation*

$$P_\Lambda(f, -s \log |a|) = 0; \tag{1}$$

(2) *the s-dimensional Hausdorff measure of Λ is positive and finite; moreover, it is equivalent to the measure m, which is the unique equilibrium measure corresponding to the Hölder continuous function $-s \log |a(x)|$ on M;*

(3) $\dim_H \Lambda = s = \dim_H m$, *in other words, the measure m is an invariant measure of full Hausdorff dimension.*

The pressure formula (1) was first obtained by Bowen [6] in a particular case. In [33], Ruelle proved that the Hausdorff dimension of a conformal repeller Λ of a $C^{1+\alpha}$ map is given by the root of Bowen's equation, and he also showed that the s-dimensional Hausdorff measure of Λ is positive and finite. In [12], Falconer showed that the Hausdorff and box dimensions of Λ coincide. For a C^1 conformal expanding map f with a conformal repeller Λ, Statements 1 and 3 of Theorem 2.1 still hold, as shown by Gatzouras and Peres in [14], and Statement 2 holds as well under some additional requirements. One can also derive Theorem 2.1 from a more general result on continuous weakly-conformal expanding maps, see [3], [30].

3. Special Classes of Non-Conformal Repellers

Unlike the case of conformal repellers the dimension theory of non-conformal repellers is in its initial stage of development, and many basic problems remain open. In this section, we describe some special classes of non-conformal repellers and focus on the following questions:

(1) Finding formulae (explicit or implicit) for Hausdorff and box dimensions of some specific repellers and Hausdorff dimension of certain Borel finite measures supported on repellers.
(2) Establishing coincidence of Hausdorff and box dimensions.
(3) Proving validity of the variational principle for Hausdorff dimension, i.e.,

$$\dim_H\Lambda = \sup\{\dim_H\mu : \mathrm{supp}(\mu) \subset \Lambda,\ \mu \text{ is } f\text{-invariant and ergodic}\},$$

where Λ is the repeller associated to the expanding map f.
(4) Showing existence of invariant measures of full Hausdorff dimension and studying their ergodic properties (this is not trivial, since the map $\mu \mapsto \dim_H\mu$ is, in general, not upper-semicontinuous).
(5) Proving finiteness of the s-dimensional Hausdorff measure of Λ (where $s = \dim_H\Lambda$).

As stated in Theorem 2.1, for conformal repellers the answers to all the above questions are affirmative. We shall see below that there are some examples of non-conformal repellers, which do not have some of the above properties. We however believe that generically they hold. To be precise, we state the following conjecture.

Conjecture 3.1. *Let $f_a : M \to M$, $a \in (\alpha, \beta)$, be a family of smooth expanding maps and let Λ_a be a repeller for f_a. If the family f_a is "typical" in a sense, then there exists a subset $A \subset (\alpha, \beta)$ of positive Lebesgue measure such that for every $a \in A$,*

(1) $\dim_H\Lambda_a = \underline{\dim}_B\Lambda_a = \overline{\dim}_B\Lambda_a = s(a)$;
(2) *there exists an f_a-invariant ergodic measure μ_a supported on Λ_a such that* $\dim_H\Lambda_a = \dim_H\mu_a$;
(3) $0 < \mathcal{H}^{s(a)}(\Lambda_a) < \infty$.

3.1. **Generalized Sierpiński carpets.** Many interesting classes of non-conformal repellers can be defined by *iterated function systems*. Before describing some specific examples we introduce some necessary definitions.

Let $\mathcal{S} = \{S_1, S_2, \cdots, S_r\}$ be a family of smooth contracting maps of the Euclidean space $\mathbb{R}^n$ that is

$$d(S_i(x), S_i(y)) \leq c_i d(x, y) \text{ for all } x, y \in X,$$

where $0 < c_i < 1$ for each i. The following statement is a well-known result of Hutchinson [16].

Theorem 3.2. *There exists a unique non-empty compact subset $\Lambda \subset X$ such that*

$$\Lambda = \bigcup_{i=1}^{r} S_i(\Lambda).$$

Moreover, let $S_{i_1 \dots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}$ for $(i_1 \cdots i_k) \in \{1, \dots, r\}^k$, then

$$\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{(i_1 \dots i_k)} S_{i_1 \cdots i_k}(E)$$

for every non-empty compact set $E \subset X$ satisfying $S_i(E) \subset E$ for all i.

Many interesting fractals arise in this way. Furthermore, let us assume that the family $\mathcal{S}$ satisfies the *open set condition* that is there exists a non-empty open set U such that $S_i(U) \subset U$, and $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$. In the case when U is a neighborhood of Λ we define a smooth expanding map $f : \bigcup_i S_i(U) \to U$ by setting $f|_{S_i(U)} = S_i^{-1}$ so that the set Λ is exactly the repeller for f.

We consider the particular case when $\{S_i\}_{i=1}^r$ are contracting affine maps that is $S_i(x) = T_i x + a_i$, where $T_i \in GL(d, \mathbb{R})$ satisfy $\|T_i\| < 1$, $a_i \in \mathbb{R}^d$. The limit set Λ generated by $\{S_1, \cdots, S_r\}$ is called a *self-affine* set. There is a special class of self-affine sets called *self-similar sets* for which $T_i = \rho_i R_i$ with $0 < \rho_i < 1$ and $R_i \in O(d)$ (in other words, $|S_i(x) - S_i(y)| = \rho_i |x - y|$, for all $x, y \in X$). In fact, if the open set condition holds, a self-similar set is actually a conformal repeller. See [8] for the dimension results of self-similar sets.

3.1.1. *Generalized Sierpiński carpets.* The study of dimensions of self-affine sets originated in the work of Bedford [5] and McMullen [26]. They considered independently a simple but nontrivial class of planar non-conformal self-affine sets and found similar formulae for the dimensions using different methods. Here we describe the approach by McMullen. In [26], he considered affine maps of the following form

$$S_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k_i/n \\ l_i/m \end{pmatrix}, \quad i = 1, \cdots, r,$$

where $1 < m \leq n$ are integers and

$$I = \{(k_i, l_i)\}_{i=1}^r \subset \{0, 1, \ldots, n-1\} \times \{0, 1, \cdots, m-1\}.$$

The self-affine set Λ generated by $\{S_i\}_{i=1}^r$ is called a *generalized Sierpiński carpet*. There is another convenient way to represent this self-affine set:

$$\Lambda = \left\{ \sum_{j=1}^{\infty} \begin{pmatrix} n^{-j} & 0 \\ 0 & m^{-j} \end{pmatrix} \begin{pmatrix} k_{i_j} \\ l_{i_j} \end{pmatrix} : \ (i_1 i_2 \cdots) \in Q \right\},$$

where $Q = \{1, \cdots, r\}^{\mathbb{N}}$. Hence we have a coding map $\chi : Q \to \Lambda$ given by

$$\chi(i_1 i_2 \cdots) = (\sum_{j=1}^{\infty} k_{i_j} n^{-j}, \sum_{j=1}^{\infty} l_{i_j} m^{-j}).$$

It is clear that Λ is an invariant repeller for the expanding toral endomorphism $f(x, y) = (nx, my)$.

Using an elementary probability method, McMullen found some formulae for Hausdorff and box dimensions of generalized Sierpiński carpets.

Theorem 3.3. *Let Λ be a generalized Sierpiński carpet defined as above. Then*

(1) *the Hausdorff dimension of Λ is given by*

$$\dim_H \Lambda = \log_m (\sum_{l=0}^{m-1} t_\ell^{\log_n m}),$$

where t_ℓ is the number of those k such that $(k, \ell) \in I$.

(2) *the box dimension of Λ is given by*

$$\dim_B \Lambda = \log_m s + \log_n (\frac{r}{s}),$$

where s is the number of those ℓ such that $(k, \ell) \in I$ for some k.

An easy calculation shows that $\dim_H \Lambda = \dim_B \Lambda$ only in some exceptional situations when (i) $n = m$ (f is conformal) and (ii) t_ℓ takes on only one value other than zero; in this case one says that I has *uniform horizontal fibres*.

To prove this result McMullen chose an optimal cover, which consists of *approximate squares*, that is rectangles of the form

$$R_u(k, l) = \left[\frac{k}{n^v}, \frac{k+1}{n^v} \right] \times \left[\frac{l}{m^u}, \frac{l+1}{m^u} \right],$$

where $v = [u \log_n m]$. Since $\mathrm{diam} R_u(k,l) \simeq m^{-u}$, the set $R_u(k,l)$ is approximately a square. If $\mathcal{U}$ is a cover of Λ by approximate squares, let $N_u(\mathcal{U})$ be the number of rectangles $R_{u'}(k,l) \in \mathcal{U}$ with $u' = u$. The role of approximate squares in computing dimension can be seen from the following statement.

Lemma 3.4. *The s-dimensional Hausdorff measure $\mathcal{H}^s(\Lambda) = 0$ if and only if for any $\varepsilon > 0$ there exists a cover $\mathcal{U}$ of Λ by approximate squares such that*

$$\sum_{u=1}^{\infty} N_u(\mathcal{U}) m^{-su} < \varepsilon.$$

McMullen also showed that there is a Bernoulli measure μ on Λ of full dimension.

3.1.2. *Self-affine sets generated by diagonal piecewise-linear maps.* In [13], Gatzouras and Lalley considered a certain class of self-affine sets, which are more general than the Sierpiński carpets studied by McMullen. Specifically, those affine mappings S_{ij} are given as

$$S_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{ij} & 0 \\ 0 & b_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_{ij} \\ d_i \end{pmatrix}, \quad (i,j) \in \mathcal{J},$$

where $\mathcal{J} = \{(i,j) : 1 \le i \le m,\ 1 \le j \le n_i\}$ is a finite index set. Here we assume that $0 < a_{ij} < b_i < 1$ and that $\sum_{i=1}^m b_i \le 1$ and $\sum_{j=1}^{n_i} a_{ij} \le 1$ for each i. Also, $0 \le d_1 < d_2 < \cdots < d_m < 1$ with $d_{i+1} - d_i \ge b_i$ and $1 - d_m \ge b_m$ and for each i, $0 \le c_{i_1} < c_{i_2} < \cdots < c_{in_i} < 1$ with $c_{i(j+1)} - c_{ij} \ge a_{ij}$ and $1 - c_{in_i} \ge a_{in_i}$.

All these requirements are needed to guarantee that the open sets $R_{ij} = S_{ij}((0,1)^2)$ are pairwise disjoint rectangles with edges parallel to the x- and y-axes arranged in rows of height b_i with height greater than width.

The self-affine set Λ determined by $\{S_{ij}\}_{(i,j)\in\mathcal{J}}$ can be coded in a natural way by a coding map $\chi : \mathcal{J}^{\mathbb{N}} \to \Lambda$ defined as follows: any $((i_1,j_1)(i_2,j_2)\cdots) \in \mathcal{J}^{\mathbb{N}}$ corresponds to the point

$$\bigcap_{k=1}^{\infty} S_{i_k j_k} \circ \cdots \circ S_{i_1 j_1}([0,1]^2).$$

Moreover, Λ is a repeller for a certain piecewise-linear expanding map $f : \mathbb{R}^2 \to \mathbb{R}^2$, which maps each R_{ij} onto $(0,1)^2$ in such a way that $f|_{R_{ij}} = S_{ij}^{-1}$. In fact, if the closed rectangles $\overline{R_{ij}}$ are disjoint, one can make f to be C^∞. Clearly, generalized Sierpiński carpets are a special case with $b_i = m^{-1}$ and $a_{ij} = n^{-1}$.

Gatzouras and Lalley obtained implicit formulae for Hausdorff and box dimensions of Λ and also established some equivalent conditions, which guarantee coincidence of Hausdorff and box dimensions.

Theorem 3.5. *The following statements hold.*

(1) *Let $p = \{p_{ij}\}$ be a probability vector on $\mathcal{J}$ and μ_p the corresponding Bernoulli measure on $\mathcal{J}^{\mathbb{N}}$. Consider the projection $\mu_{p*} = \chi_* \mu_p$ on Λ. Then*
$$\begin{aligned}\dim_H \mu_{p*} = &\frac{\sum\sum p_{ij}\log p_{ij}}{\sum\sum p_{ij}\log a_{ij}} \\ &+ \sum q_i \log q_i \left(\frac{1}{\sum q_i \log b_i} - \frac{1}{\sum\sum p_{ij}\log a_{ij}}\right),\end{aligned}$$
where $q_i = \sum_j p_{ij}$.

(2) $\dim_H \Lambda = s = \max\{\dim_H \mu_{p*}\}$, *where the maximum is taken over all probability vectors p on $\mathcal{J}$.*

(3) $\dim_B \Lambda = \delta$ *is the unique root of the equation*
$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} b_i^t a_{ij}^{\delta - t} = 1,$$
where t is the unique root of the equation $\sum\limits_{i=1}^{m} b_i^t = 1$.

Theorem 3.6. *The following statements are equivalent:*

(1) $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$;
(2) $0 < \mathcal{H}^s(\Lambda) < \infty$;
(3) $\sum\limits_j a_{ij}^{s-t} = 1$ *for all* $i = 1, \cdots, m$.

Theorem 3.5 shows that there is a Bernoulli measure of full dimension, however, no explicit formula for the Hausdorff dimension is presented. Theorem 3.6 implies that, similar to generalized Sierpiński carpets, coincidence of Hausdorff dimension and box dimensions is highly *non-typical*, since the equalities in Statement 3 of Theorem 3.6 rarely hold.

It follows from Statement 2 of Theorem 3.6 that the s-dimensional Hausdorff measure of Λ is in general either zero or infinite. Later in [29], Peres has shown that for a typical generalized Sierpiński carpet (i.e. neither conformal nor having uniform horizontal fibres) the s-dimensional Hausdorff measure must be infinite.

Observe that the basic sets in the construction by Gatzouras and Lalley are rectangles $R_{ij} = S_{ij}((0,1)^2)$. In [2], Barański studied a more general class of planar geometric constructions for which the basic sets are *rectangle-like*. He obtained some results on the Hausdorff and box dimensions of the limit sets for these constructions, which are similar to Theorem 3.5. It is worth mentioning that this class includes some non-affine limit sets, such as the *flexed Sierpiński gasket*.

3.1.3. *Self-affine Sierpiński sponges.* Another version of Sierpiński carpets was considered by Kenyon and Peres in [18]. They extended the results of McMullen and Bedford in two directions: to higher dimensions and to arbitrary invariant sets.

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a linear expanding endomorphism, which is the direct product of conformal endomorphisms of $\mathbb{T}$. More precisely, given a $d \times d$ diagonal matrix $M = \mathrm{diag}\{m_\nu\}_{\nu=1}^d$, where $1 < m_1 < \cdots < m_d$ are integers, we define an expanding transformation on $\mathbb{T}^d$ by $f(x) = Mx$. Choose an index set

$$I = \{a_1, \cdots, a_r\} \subset \prod_{\nu=1}^{d} \{0, 1, \ldots, m_\nu - 1\}$$

and consider the compact f-invariant subset of $\mathbb{T}^d$

$$\Lambda = \left\{ \sum_{j=1}^{\infty} M^{-j} a_{i_j} : \ (i_1 i_2 \cdots) \in Q \right\},$$

where $Q = \{1, \cdots, r\}^{\mathbb{N}}$. There is a natural coding map $\chi : Q \to \Lambda$ given by $\chi(i_1 i_2 \cdots) = \sum_{j=1}^{\infty} M^{-j} a_{i_j}$.

Λ can also be obtained as the limit set for the construction given by the family of contracting affine maps $S_i(x) = M^{-1}x + b_i$, $i = 1, \cdots, r$, where $b_i = (a_i^1/m_1, \cdots, a_i^d/m_d)$ and $a_i = (a_i^1, \cdots, a_i^d) \in I$. This justifies calling Λ a *self-affine Sierpiński sponge*. Two-dimensional self-affine Sierpiński sponges are exactly generalized Sierpiński carpets.

Kenyon and Peres obtained some formulae for Hausdorff and box dimensions of self-affine Sierpiński sponges. To state their result define a sequence of partition functions Z_k of k arguments on $\prod_{\nu=1}^{k} \{0, 1, \ldots, m_\nu - 1\}$, $0 \le k \le d$, as follows. If Z_d is the indicator of I then set inductively

$$Z_{k-1}(j_1, \cdots, j_{k-1}) = \sum_{j_k=0}^{m_k - 1} Z_k(j_1, \cdots, j_k)^{r_k},$$

where $r_d = 1$ and $r_k = \frac{\log m_k}{\log m_{k+1}}$ for $1 \le k < d$.

Theorem 3.7. *The following statements hold:*

(1) *The Hausdorff dimension*

$$\dim_H \Lambda = \frac{\log Z_0}{\log m_1}.$$

Moreover, Λ supports a unique ergodic invariant probability measure of full Hausdorff dimension.

(2) *The box dimension*

$$\dim_B \Lambda = \sum_{\nu=1}^{d} \frac{1}{\log m_\nu} \log \frac{\mathrm{Card}(\pi_\nu(I))}{\mathrm{Card}(\pi_{\nu-1}(I))},$$

where π_ν is the projection to the first ν coordinates, i.e., $\pi_\nu(j_1, \cdots, j_d) = (j_1, \cdots, j_\nu)$, and by convention $\mathrm{Card}(\pi_0(I)) = 1$.

In addition, one can obtain necessary and sufficient conditions for the coincidence of the Hausdorff and box dimensions, which is similar to the uniform horizontal fibres condition in the case of generalized Sierpiński carpets.

Theorem 3.8. $\dim_H \Lambda = \dim_B \Lambda$ *if and only if for each $1 \le \nu < d$ the cardinality of the preimages $\pi_\nu^{-1}(i_1, \cdots, i_\nu)$ are the same for all elements $(i_1, \cdots, i_\nu)$ of $\pi_\nu(I)$.*

The dimension results for self-affine Sierpiński sponges stated above are quite similar to those for generalized Sierpinski carpets studied by Bedford and McMullen, but require a more sophisticated techniques.

A remarkable corollary of the above results is that for *any* compact f-invariant subset of $\mathbb{T}^d$, which is not necessarily a self-affine Sierpiński sponge, there exists an f-invariant measure of full Hausdorff dimension. To see this let F be a compact f-invariant subset of $\mathbb{T}^d$. We shall approximate it by a sequence $\{\Lambda_N\}_{N=1}^{\infty}$ of self-affine Sierpiński sponges. To do this choose a suitable index set

$$I = \{a_i\}_{i=1}^{r} \subset \prod_{\nu=1}^{d} \{0, 1, \dots, m_\nu - 1\}$$

such that the correspondent Sierpiński sponge Λ_0 contains F (note that Λ_0 may be the whole d-torus). Let $Q = \{1, \dots, r\}^{\mathbb{N}}$ and $\chi : Q \to \Lambda_0$ be the coding map. Let also $J = \chi^{-1}(F) \subset Q$ and $J^{(N)}$ ($N \in \mathbb{N}$) be the collection of sequences $\{w_j\} \in Q$ such that for every $q \ge 0$ the word $(w_{qN+1}, \cdots, w_{qN+N})$ can be extended on the right to a sequence in J. The set $\Lambda_N = \chi(J^{(N)})$ is a self-affine Sierpiński sponge with respect to f^N, and according to Theorem 3.7, there exists an ergodic f^N-invariant probability measure μ_N of full Hausdorff dimension supported on Λ_N. Define $\mu_N^* = \frac{1}{N} \sum_{k=0}^{N-1} \mu_N \circ f^{-k}$. Then μ_N^* is an f-invariant probability measure supported on $\cup_{k=0}^{N-1} f^k \Lambda_N$. Passing to a weak*-limit μ of $\{\mu_N^*\}$, Kenyon and Peres proved that there exists an ergodic component $\tilde{\mu}$ of μ supported on F for which $\dim_H \tilde{\mu} = \dim_H F$.

Using the Ledrappier-Young dimension formula (see [20, 21]) one can establish uniqueness of the ergodic measure of full dimension for the self-affine

Sierpiński sponge Λ. Indeed, for a smooth expanding map f of a surface with globally defined strong and weak unstable foliations, Ledrappier and Young showed that the Hausdorff dimension of an ergodic measure μ is given by the formula

$$\dim_H \mu = \frac{1}{\lambda^1_\mu(f)} h_\mu(f) + \left(\frac{1}{\lambda^2_\mu(f)} - \frac{1}{\lambda^1_\mu(f)} \right) h_{\pi_*\mu}(f_*), \tag{2}$$

where f_* is the action of f on the leaves of the strong unstable foliation, $\pi_*\mu$ the projection of μ on the space of leaves and $0 < \lambda^2_\mu(f) < \lambda^1_\mu(f)$ are the Lyapunov exponents of f with respect to μ. Applying this to the expanding map $f(x,y) = (nx, my)$ on $\mathbb{T}^2$ $(n \geq m)$, we have $\pi(x,y) = y$, $f_*(y) = my$ and hence,

$$\dim_H \mu = \frac{1}{\log n} h_\mu(f) + \left(\frac{1}{\log m} - \frac{1}{\log n} \right) h_{\pi_*\mu}(f_*). \tag{3}$$

Moreover, consider the map $f(x) = Mx$ on $\mathbb{T}^d$ where $M = \mathrm{diag}\{m_\nu\}_{\nu=1}^d$ and $1 < m_1 < \cdots < m_d$ are integers. Building upon the special diagonal structure of f, Kenyon and Peres extended (3): for an ergodic invariant measure μ supported on $\mathbb{T}^d$,

$$\dim_H \mu = \frac{1}{\log m_d} h_\mu(f) + \sum_{\nu=1}^{d-1} \left(\frac{1}{\log m_{\nu-1}} - \frac{1}{\log m_\nu} \right) h_{\pi_{\nu*}\mu}(f_\nu),$$

where $\pi_\nu(j_1, \cdots, j_d) = (j_1, \cdots, j_\nu)$ and $f_\nu(j_1, \cdots, j_\nu) = (m_1 j_1, \cdots, m_\nu j_\nu)$.

Let us stress that for an arbitrary compact invariant subset $F \subset \mathbb{T}^d$ only the existence of an f-invariant measure of full dimension supported on F was proved in [18]. Uniqueness of such a measure is a more difficult problem. In [19], Kenyon and Peres determined the Hausdorff and box dimensions for the *sofic affine-invariant sets* of $\mathbb{T}^2$, that is the compact invariant subsets of $\mathbb{T}^2$ corresponding to some sofic systems of the full shift Q. Recently in [27], Olivier has discussed the uniqueness of the invariant measure of full dimension for the sofic affine-invariant sets of $\mathbb{T}^2$, and established uniqueness under some extra conditions.

3.1.4. *Perturbed generalized Sierpiński carpets.* In [22],[24], Luzia considered a class of skew-product expanding maps of the 2-torus, which are a C^2-perturbation of a generalized Sierpiński carpet.

More precisely, let $f_0 : \mathbb{T}^2 \to \mathbb{T}^2$ be given by $f_0(x,y) = (nx, my)$ $(n > m)$ and $I \subset \{(i,j) : 0 \leq i < n,\ 0 \leq j < m\}$ be an index set. Let also Λ_0 be a generalized Sierpiński carpet generated by (f_0, I). Since f_0 is expanding, there exists $\varepsilon > 0$ such that if a map f is ε-close to f_0 in the C^1 topology, then there is a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ close to the identity which conjugates f and

f_0, i.e., $f \circ h = h \circ f_0$. We call the f-invariant set $\Lambda = h(\Lambda_0)$ an *f-continuation* of Λ_0.

We consider the class $\mathfrak{F}$ of C^2 skew-product maps $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the form $f(x, y) = (a(x, y), b(y))$. For $f \in \mathfrak{F}$ the set Λ can be constructed geometrically as a generalized Sierpiński carpet using a distorted grid of lines. It is easy to see that f preserves horizontal lines, while distorting vertical lines to C^2 curves. Thus there are m horizontal lines and n distorted C^2 vertical curves that divide $\mathbb{T}^2$ into $n \times m$ distorted rectangles, each of which is mapped by f onto the entire $\mathbb{T}^2$. After choosing an index set $I \subset \{0, \cdots, n-1\} \times \{0, \cdots, m-1\}$, where each $(i, j) \in I$ corresponds to a distorted rectangle R_{ij}, we define the contracting map $S_{(i,j)} = (f|_{R_{ij}})^{-1}$. One can easily check that the f-continuation Λ is precisely the limit set generated by $\{S_{(i,j)}\}_{(i,j)\in I}$.

The conjugacy map h and its inverse are only Hölder-continuous with exponent $(\log m - \varepsilon)/(\log n + \varepsilon)$. Therefore the problem of calculating the Hausdorff dimension in this setting cannot be trivially reduced to the linear case, namely, $\dim_H \Lambda \neq \dim_H \Lambda_0$ in general. The following result provides a characterization of the Hausdorff dimension in terms of the variational principle.

Theorem 3.9. *Let Λ_0 be a generalized Sierpiński carpet generated by (f_0, I). Suppose $f \in \mathfrak{F}$ is ε-close to f_0 in the C^2 topology and Λ is the f-continuation of Λ_0. Then*

(1) *the variational principle for the Hausdorff dimension holds,*
$$\dim_H \Lambda = \sup\{\dim_H \mu : \mu(\Lambda) = 1, \\ \mu \text{ is } f\text{-invariant and ergodic}\};$$
(2) *there exists an ergodic invariant measure μ on Λ of full dimension;*
(3) *the map $B_{C^2}(f_0, \varepsilon) \cap \mathfrak{F} \ni f \mapsto \dim_H \Lambda_f$ is continuous, where $B_{C^2}(f_0, \varepsilon)$ is the ball in the C^2-topology, centered at f_0 of radius ε, and Λ_f is the f-continuation of Λ_0.*

The idea for proving the variational principle for the Hausdorff dimension is derived from [13] and it is to approximate Λ by self-affine sets Λ_n, which are generated by the self-affine action of f^n on each basic rectangles of order n and then to show that due to the bounded distortion property the error term vanishes as n increases.

Statement 3 of the theorem may not be true outside of the class $\mathfrak{F}$ even when one considers a smoothly parametrized family of fractal sets, as shown in [13].

The proof of existence of the measure of full Hausdorff dimension is based on the following statement known as the *relativized variational principle.*

Theorem 3.10. *Let (X,T) and (Y,S) be mixing subshifts of finite type and S a factor of T with the factor map π. Let $\varphi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be positive Hölder continuous functions. Then the maximum of*

$$\frac{h_{\mu\circ\pi^{-1}}(S)}{\int \psi\circ\pi d\mu} + \frac{h_\mu(T) - h_{\mu\circ\pi^{-1}}(S)}{\int \varphi d\mu} \tag{4}$$

over all T-invariant Borel probability measures on X is attained on the subset of ergodic measures.

For the notions of subshifts of finite type and their factors we refer the reader to [30]. To apply this result to the f-continuation Λ of a generalized Sierpiński carpet Λ_0 we let $\pi : \mathbb{T}^2 \to \mathbb{T}^1$ be the projection given by $\pi(x,y) = y$. Since $f(x,y) = (a(x,y), b(y))$, we obtain that $\pi\circ f = b\circ\pi$. Let $T \equiv f|_\Lambda$, $S \equiv b|_{\pi\Lambda}$, $\varphi \equiv \log\partial_x a$ and $\psi \equiv \log b'$. (We use "$\equiv$" instead of "$=$" to stress that these equalities hold up to the conjugacy between the maps and its symbolic representation.)

Given an ergodic f-invariant measure μ supported on Λ, let $\nu = \mu\circ\pi^{-1}$. Applying the Ledrappier-Young dimension formula (2), we have

$$\dim_H\mu = \frac{h_\nu(b)}{\int \log b' \ d\nu} + \frac{h_\mu(f) - h_\nu(b)}{\int \log\partial_x a \ d\mu},$$

which is exactly of the form (4). Note that since f is a small perturbation of $f_0(x,y) = (nx, my)$ with $n > m$, we have $\inf|\partial_x a| > \sup|b'|$ and the Lyapunov exponents are

$$\lambda^1_\mu(f) = \int \log\partial_x a \ d\mu > \int \log b' \ d\nu = \lambda^2_\mu(f).$$

According to Statement 1 of Theorem 3.9, $\dim_H\Lambda = \sup\dim_H\mu$, where the supremum is taken over all ergodic f-invariant measure supported on Λ. By Theorem 3.10, this supremum can actually be obtained.

In particular, one has that the measure of full Hausdorff dimension is an equilibrium measure for a relativized variational principle. Theorem 3.10 can also be used to establish existence of invariant ergodic measures of full dimension for products of two conformal expanding maps (see [14] and Section 3.3).

3.2. **Almost every type results for general self-affine sets.** As we have seen above the Hausdorff and box dimensions of conformal self-affine sets coincide and the Hausdorff measure at dimension is strictly positive and finite. On the other hand, the structure of general self-affine sets can be quite complicated, their Hausdorff and box dimensions may not agree and the Hausdorff measure at dimension need not be strictly positive or finite. It is however believed that for "typical" self-affine sets the situation is not so bad.

In [9], Falconer established a fundamental result for general self-affine sets. Consider affine maps $S_i(x) = T_i x + a_i$ where $T_i \in GL(d, \mathbb{R})$ are contractions and $a_i \in \mathbb{R}^d$ translation vectors, $1 \le i \le r$. Let Λ be the self-affine set determined by the maps $S_1, \cdots, S_r$. It turns out that the Hausdorff and box dimensions of Λ are related to the *singular value functions* defined as follows. Given any nonsingular contracting linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$, the *singular values* α_i ($1 \le i \le d$) of T are the lengths of the principle semiaxes of $T(B)$, where B is the unit ball in $\mathbb{R}^d$. Equivalently, the singular values are the eigenvalues of $\sqrt{T^*T}$ (where T^* is the transpose of T). Assume that $1 > \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_d > 0$ and define the *singular value function* by

$$\phi^s(T) = \begin{cases} \alpha_1\alpha_2\cdots\alpha_{\lfloor s\rfloor}\alpha_{\lfloor s\rfloor+1}^{s-\lfloor s\rfloor}, & 0 \le s < d \\ \\ (\alpha_1\alpha_2\cdots\alpha_d)^{s/d}\ , & s \ge d \end{cases} \tag{5}$$

Note that for every $s \ge 0$ the function ϕ^s is sub-multiplicative, i.e.,

$$\phi^s(TU) \le \phi^s(T)\phi^s(U), \quad T, U \in GL(d, \mathbb{R}).$$

Write $Q = \{1, \ldots, r\}^{\mathbb{N}}$, $Q_k = \{1, \ldots, r\}^k$ for $k \in \mathbb{N}$, and $T_{\mathbf{i}} = T_{i_1} \cdots T_{i_k}$ for $\mathbf{i} = (i_1 \ldots i_k) \in Q_k$.

Proposition 3.11. *There exists a unique number $s > 0$ such that*

$$\lim_{k\to\infty}[\sum_{\mathbf{i}\in Q_k}\phi^s(T_{\mathbf{i}})]^{1/k} = 1$$

and

$$s = \inf\{t : \sum_{k=1}^{\infty}\sum_{\mathbf{i}\in Q_k}\phi^t(T_{\mathbf{i}}) < \infty\} = \sup\{t : \sum_{k=1}^{\infty}\sum_{\mathbf{i}\in Q_k}\phi^s(T_{\mathbf{i}}) = \infty\}.$$

The number s is called *Falconer's dimension* and is denoted by $d(T_1, \ldots, T_r)$.

Let us fix the linear parts T_i of the affine maps S_i, $i = 1, \ldots, r$ and denote by $\Lambda(\mathbf{a})$ the self-affine sets generated by $\{S_i\}_{i=1}^r$ where $\mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{R}^{rd}$ is the translation vector. We also set $\eta = \max_{1\le i\le r}\|T_i\|$.

Theorem 3.12. *Assume that $\eta < \frac{1}{3}$. Then for almost all $\mathbf{a} \in \mathbb{R}^{rd}$ (in the sense of rd-dimensional Lebesgue measure) we have*

$$\dim_H\Lambda(\mathbf{a}) = \dim_B\Lambda(\mathbf{a}) = \min(d, d(T_1, \cdots, T_r)).$$

In [17] Käenmäki has shown that for almost all $\mathbf{a} \in \mathbb{R}^{rd}$ the set $\Lambda(\mathbf{a})$ supports an ergodic measure of full dimension.

Note that Falconer's dimension $d(T_1, \ldots, T_r)$ does not depend on the choice of $\mathbf{a} \in \mathbb{R}^{rd}$, i.e., the Hausdorff and box dimensions of a typical self-affine set do not depend on the locations of basic sets.

The assumption $\eta < \frac{1}{3}$ is needed only to obtain the lower bound for the Hausdorff dimension.

Proposition 3.13. *For all* $\mathbf{a} \in \mathbb{R}^{rd}$*, we have*

$$\dim_H \Lambda(\mathbf{a}) \leq \underline{\dim}_B \Lambda(\mathbf{a}) \leq \overline{\dim}_B \Lambda(\mathbf{a}) \leq \min(d, d(T_1, \cdots, T_r)).$$

3.2.1. *Number-theoretical peculiarities in the case* $\eta > \frac{1}{2}$. Although generic self-affine sets have the Hausdorff and box dimensions equal to each other some other interesting questions remain open. Among them is finding an optimal upper bound for the number η. In [34], Solomyak showed by a slight modification of Falconer's arguments that the upper bound can be improved to $\frac{1}{2}$. However, when $\eta > \frac{1}{2}$, some obstructions occur, related to certain number-theoretical peculiarities. To describe this phenomenon we begin with an example of self-affine sets discussed by Pollicott and Weiss in [32].

Let $R_1, R_2 \subset [0,1]^2$ be two disjoint rectangles in the unit square, aligned with the axes, each having the same height $0 < \lambda_1 < 1$ and width $0 < \lambda_2 < 1$. For convenience, we assume that $0 < \lambda_1 \leq \lambda_2 < 1$ and $\lambda_1 \leq \frac{1}{2}$. Now consider two affine maps $S_1 : [0,1]^2 \to R_1$ and $S_2 : [0,1]^2 \to R_2$, which contract the unit square by λ_1 in the vertical direction and by λ_2 in the horizontal direction. Finally, let Λ denote the self-affine set generated by S_1 and S_2.

In the very special "degenerate" case when the rectangle R_1 lies directly above or below the rectangle R_2 the limit set Λ is just a one-dimensional Cantor set in the vertical direction. In what follows we consider the non-degenerate case.

Exploiting the idea of approximate squares, Pollicott and Weiss obtained the following results on the Hausdorff and box dimensions of Λ. First, they found a formula for the box dimension.

Theorem 3.14. *The following formula holds*

$$\dim_B \Lambda = \begin{cases} -\frac{\log 2}{\log \lambda_2} & \text{if } 0 < \lambda_2 \leq \frac{1}{2}, \\ -\log \frac{2\lambda_2}{\lambda_1} / \log \lambda_1 & \text{if } \frac{1}{2} \leq \lambda_2 < 1. \end{cases}$$

In the case $0 < \lambda_2 < \frac{1}{2}$ they proved the following result about the Hausdorff dimension.

Theorem 3.15.
$$\dim_H \Lambda = \dim_B \Lambda = -\frac{\log 2}{\log \lambda_2}.$$

When $\frac{1}{2} < \lambda_2 < 1$, certain number-theoretical properties of the number λ_2 plays an important role in computing the Hausdorff dimension. To this end recall that a real number $\beta \in [0, 1]$ is a *GE number* (after Garsia and Erdös) if there exists a constant $C > 0$ such that for all $x \in [0, \infty)$,

$$\text{Card}\left\{(i_0, \cdots, i_{n-1}) \in \{0,1\}^n : \sum_{r=0}^{n-1} i_r \beta^r \in [x, x + \beta^n)\right\} \leq C(2\beta)^n.$$

It is an interesting problem to determine whether a given number is a GE number, see [35]. It is known that

(1) there is no GE number in the interval $(0, \frac{1}{2})$;
(2) almost all numbers in $(\frac{1}{2}, 1)$ are GE numbers with respect to the Lebesgue measure [36];
(3) there exist numbers in $(\frac{1}{2}, 1)$ that are not GE numbers; for example, $\frac{\sqrt{5}-1}{2}$; indeed, any reciprocal of a PV-number is not a GE number (a PV number, after Pisot and Vijayarghavan, is a root of an algebraic equation whose all conjugates have moduli less than one).

Theorem 3.16. *The following statements hold:*

(1) *If* $\frac{1}{2} < \lambda_2 < 1$, *then*
$$\dim_B \Lambda = -\log \frac{2\lambda_2}{\lambda_1} / \log \lambda_1.$$
(2) *If* $\frac{1}{2} < \lambda_2 < 1$ *is a GE number, then*
$$\dim_H \Lambda = \dim_B \Lambda = -\log \frac{2\lambda_2}{\lambda_1} / \log \lambda_1.$$

It follows that for almost every $\lambda_2 \in (0, 1)$, the Hausdorff dimension and box dimension coincide. Furthermore, the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure on Λ is the measure of full Hausdorff dimension, since in this case the two basic rectangles R_1 and R_2 are of the same size.

For exceptional values of λ_2 (for example when it is the reciprocal of a PV-number), the dimension drop occurs, i.e., the Hausdorff dimension becomes strictly less than the box dimension. This follows from the work of Przytycki and Urbański [31] as observed by Edgar [7]: take $\lambda_1 = \frac{1}{2}$ and λ_2 to be a reciprocal of a PV-number, then $\dim_B \Lambda = 2 + \frac{\log \lambda_2}{\log 2}$, while $\dim_H \Lambda < 2 + \frac{\log \lambda_2}{\log 2}$ for some suitable choice of translation vectors for S_1 and S_2. Moreover, in this case the linear parts of both S_1 and S_2 is $\text{diag}(\frac{1}{2}, \lambda_2)$, and hence Falconer's dimension is equal to $2+\frac{\log\lambda_2}{\log 2}$, which is greater than the value of the Hausdorff dimension. Since there is a sequence of PV-numbers approaching 2 from below, Falconer's dimension formula holds only under the condition $\eta<\frac{1}{2}$. Let us stress that examples, when Falconer's dimension formula fails, are known only in the case $d=2$. It seems plausible that if $d\ge 3$, the formula holds under the natural assumption $\eta<1$ (see [28]).

3.2.2. *Checkable conditions for the validity of Falconer's dimension formula.* Falconer's proof of the dimension formula does not provide any information as to which translation vectors $(a_1,\cdots,a_r)$ this formula applies. One would like to have some checkable conditions for the validity of Falconer's dimension formula. As soon as the box dimension is concerned such conditions were obtained by Falconer in [10]. We say that an open set $U\subset\mathbb{R}^d$ satisfies the *projection condition* if

$$\mathcal{L}^{d-1}(\mathrm{proj}_\Pi U)=\mathcal{L}^{d-1}(\mathrm{proj}_\Pi \bar U)$$

for all $(d-1)$-dimensional subspaces Π. Here $\mathcal{L}^k$ is the k-dimensional Lebesgue measure and $\bar U$ denotes the closure of the set U.

Theorem 3.17. *Let Λ be the invariant set for affine contractions $S_1,\dots,S_r$ on $\mathbb{R}^d$. Suppose that Λ satisfies the open set condition for a set U for which the projection condition holds and that for some $c>0$ we have $\mathcal{L}^{d-1}(proj_\Pi\Lambda)\ge c$ for all $(d-1)$-dimensional subspaces Π. Then $\dim_B\Lambda=d(T_1,\cdots,T_r)$.*

Checkable conditions for the Hausdorff dimension are more delicate and some progress in this direction has been made by Hueter and Lalley in [15] who used some ideas from thermodynamic formalism. They considered a self-affine set Λ in $\mathbb{R}^2$, generated by affine contractions $S_1,\dots,S_r$ and found some sufficient conditions for the validity of Falconer's dimension formula:

(HL1) (contractivity) $\eta<1$;
(HL2) (bounded distortion) $\alpha(T_i)^2<\beta(T_i)$, where $1>\alpha(T_i)\ge\beta(T_i)>0$ are singular values of T_i, $1\le i\le r$ (as defined in Section 3.2);
(HL3) (separation) if $\mathcal{Q}_2$ is the closed second quadrant of $\mathbb{R}^2$, then the sets $T_i^{-1}(\mathcal{Q}_2)$ are pairwise disjoint subsets of $\mathrm{Int}(\mathcal{Q}_2)$;
(HL4) (orientation) each T_i has positive determinant;
(HL5) (closed set conditions) there exists a bounded open set U such that the images $\overline{S_i(U)}$ are pairwise disjoint closed subsets of U.

Before we state the result, let us comment on these conditions. The key Conditions (HL2) and (HL3) are both easy to check. (HL5) is equivalent to assuming that the compact sets $S_i(\Lambda)$ are pairwise disjoint – a requirement that is a little stronger than the open set condition. Condition (HL5) implies that the coding map $\chi : \{1, \dots, r\}^{\mathbb{N}} \to \Lambda$ is a homeomorphism, and hence Λ is totally disconnected. Condition (HL2) originated in [11]. Note that Condition (HL4) is not really necessary.

Let us now state the main result.

Theorem 3.18. *Suppose that affine contractions S_i, $i = 1, \dots, r$ satisfy Conditions (HL1)-(HL5). Then*

(1) $s = \dim_H \Lambda = \dim_B \Lambda = d(T_1, \dots, T_r) < 1$, *i.e., Falconer's dimension formula holds;*
(2) *the s-dimensional Hausdorff measure of Λ is finite, i.e., $\mathcal{H}^s(\Lambda) < \infty$;*
(3) *there exists a unique ergodic invariant probability measure of full Hausdorff dimension.*

3.2.3. *An open class of repellers in $\mathbb{R}^2$.* In [15], sufficient conditions, in the setting of self-affine sets, were given for Falconer's dimension formula to be valid. In [23], Luzia extended this result to the case of nonlinear contractions.

Let $\mathcal{S} = \{S_1, \dots, S_r\}$ be a collection of C^2 diffeomorphisms satisfying the following conditions:

(L1) (contractivity) $\max\limits_{1 \le i \le r} \sup\limits_{x} \|DS_i(x)\| < 1$;
(L2) (non-overlapping) there is a convex bounded open set U such that $S_i(\overline{U})$ are pairwise disjoint subset of U;
(L3) (domination) each S_i satisfies $DS_i(x)\mathcal{P}_+ \subset \mathrm{Int}\mathcal{P}_+$ for all $x \in U$, where $\mathcal{P}_+$ is the union of the closed first and third quadrants of $\mathbb{R}^2$ (as a set of lines or a set of vectors depending on the context);
(L4) (bounded distortion) each S_i satisfies
$$\frac{|DS_i(x)v|^3}{|\det DS_i(x)|} < 1$$
for all $x \in U$ and $v \in \mathcal{P}_+$ with $|v| = 1$.

Note that Condition (L2) is almost the same as the closed set condition but with extra requirement that U is convex. Let $Q = \{1, \cdots, r\}^{\mathbb{N}}$, and define the coding map $\chi : Q \to \Lambda$ by $\chi(\mathbf{i}) = \lim_{n\to\infty} S_{\mathbf{i}|n}(\overline{U})$ where $\mathbf{i} = (i_1, i_2, \dots) \in Q$ and $S_{\mathbf{i}|n} = S_{i_1} \circ \cdots \circ S_{i_n}$. Let $\sigma : Q \to Q$ be the left shift on Q. For each i $\in Q$ set

$$V(\mathbf{i}) = \lim_{n\to\infty} DS_{\mathbf{i}|n}(S_{\mathbf{i}|n}^{-1}(\chi(\mathbf{i})))\mathcal{P}_+, \quad \varphi(\mathbf{i}) = \log |DS_{i_1}(\chi(\sigma\mathbf{i}))V(\sigma\mathbf{i})|.$$

Using Condition (L1) one can apply methods of thermodynamic formalism to the potential φ to show that there is a unique $s > 0$ such that $P(s\varphi) = 0$, where $P(\cdot)$ is the topological pressure.

Theorem 3.19. *Suppose that the contractions S_i, $i = 1, \dots, r$ satisfy Conditions (L1)-(L4). Then*

(1) $\dim_H \Lambda = \dim_B \Lambda = s$;
(2) *The s-dimensional Hausdorff measure of Λ is finite;*
(3) *There exists a unique ergodic invariant probability measure of full Hausdorff dimension.*

One can see that conditions and conclusions of Theorem 3.19 are parallel to those of Theorem 3.18. Moreover, consider the space

$$\mathcal{L} = \{(S_1, \dots, S_r) : \text{each } S_i \text{ is of class } C^2 \text{ and satisfies } (L1) - (L4)\}.$$

We endow $\mathcal{L}$ with a natural topology such that $(\tilde{S}_1, \dots, \tilde{S}_r)$ is ε-close to $(S_1, \dots, S_r)$ if $\tilde{S}_i$ is ε-close to S_i in the C^2 topology for each i. It is clear that $\mathcal{L}$ is open in this topology. Now given $\mathcal{S} \in \mathcal{L}$, let $\Lambda_{\mathcal{S}}$ denote the limit set determined by $\mathcal{S}$.

Theorem 3.20. *The function $\mathcal{L} \ni \mathcal{S} \mapsto \dim_H \Lambda_{\mathcal{S}}$ is continuous.*

This implies that for any one-parameter family $\mathcal{S}(t) \in \mathcal{L}$, the Hausdorff dimension of the limit sets generated by $\mathcal{S}(t)$ depends continuously on t.

Note that the domination and separation conditions are quite specific to the case of $\mathbb{R}^2$ and extending them to higher dimension is an interesting open problem.

3.3. **Product of two conformal expanding maps.** One of the simplest examples of non-conformal repellers is a repeller associated to the product of two conformal expanding maps. In [14], Gatzouras and Peres considered this case and established the variational principle for the Hausdorff dimension. More precisely, let $f_i : U_i \to M_i$ be a conformal expanding C^1 map of an open subset $U_i \subset M_i$ of a Riemannian manifold M_i and let Λ_i be the repeller for f_i, $i = 1, 2$. Using Markov partitions of Λ_i, one can identify each $f_i|_{\Lambda_i}$ with a subshift of finite type via the conjugacy map χ_i. Consider the direct product map $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ and the product set $\Lambda_1 \times \Lambda_2$, which could also be identified with a subshift of finite type via the conjugacy map $\chi = \chi_1 \times \chi_2$. To study the action of f on this set we need the following notion.

Let $\mathcal{L}$ be a finite alphabet. A subshift $Y \subset \mathcal{L}^{\mathbb{N}}$ satisfies the *specification property* if there exists $m \in \mathbb{N}$ such that for any two blocks ξ_1 and ξ_2 in $\cup_{n \geq 0} \mathcal{L}^n$ with ξ_i extendable to a sequence in Y, there exists a block $\zeta \in \mathcal{L}^m$ such that $\xi_1 \zeta \xi_2$ extends to a sequence in Y.

Roughly speaking, the specification property can be thought of as a quantitative version of topological mixing.

Theorem 3.21. *Consider a compact f-invariant subset $\Lambda \subset \Lambda_1 \times \Lambda_2$ where Λ need not be a Cartesian product of f_i-invariant subsets. Suppose that*

$$\min_{x_1\in\Lambda_1} |Df_1(x_1)| \geq \max_{x_2\in\Lambda_2} |Df_2(x_2)|;$$

and that $\chi^{-1}(\Lambda)$ satisfies the specification property. Then the variational principle for the Hausdorff dimension holds, i.e.,

$$\dim_H \Lambda = \sup\{\dim_H \mu : \ \mu(\Lambda) = 1, \ \mu \textit{ is } f\textit{-invariant and ergodic}\}.$$

Note that a generalized Sierpiński carpet is a special example of this type associated with the linear expanding map $f(x, y) = (nx, my)$ of $\mathbb{T}^2$, and as mentioned in Section 3.1.1, it has a Bernoulli measure of full dimension.

In the proof of this result, Gatzouras and Peres used some results from [13] as well as the idea from [18] of approximating arbitrary invariant sets by self-affine sets.

One can also compute the box dimension of Λ by the formula $\dim_B \Lambda = r+\theta$, where r and θ solve the equations

$$\begin{cases} P(-r \log |Df_2|_{\Lambda_2^*}) = 0, \\ P(-r \log |Df_2|_{\Lambda_2^*} - \theta \log |Df_1|_{\Lambda_1^*}) = 0 \end{cases}$$

with $\Lambda_i^* = \mathrm{proj}_i(\Lambda)$ and $P(\cdot)$ to be the topological pressure.

The specification property is quite strong and it is conjectured that for an expanding map f with this property and for any compact invariant subset Λ, there exists a unique ergodic invariant measure of full Hausdorff dimension.

4. Dimension Estimates

In the previous sections we studied the Hausdorff and box dimensions of non-conformal repellers with simple geometric structure such as self-affine sets or the ones, which can be approximated by simple self-affine sets. In general, the Hausdorff and box dimensions of a non-conformal repeller do not agree and one can only hope that this may happen generically and then the common value can be computed by a formula that is somewhat similar to Bowen's formula in the conformal case. We therefore describe some effective estimates from above and from below of the dimension.

4.1. **Lower bounds for self-affine sets.** As a sequel to [9], Falconer obtained a lower bound for the Hausdorff dimension of self-affine sets Λ (see [10]) by producing a number $d_-(T_1, \ldots, T_r)$, which is similar to $d(T_1, \ldots, T_r)$.

For a contracting linear map $T \in GL(d, \mathbb{R})$, take $\psi^s(T) = \phi^s(T^{-1})^{-1}$, where $\phi^s(\cdot)$ is the singular value function defined in Section 3.2. It is easy to check that ψ^s is supermultiplicative, i.e., $\psi^s(TU) \geq \psi^s(T)\psi^s(U)$. Write

$Q = \{1, \cdots, r\}^{\mathbb{N}}$, $Q_k = \{1, \cdots, r\}^k$ for $k \in \mathbb{N}$, and $T_{\mathbf{i}} = T_{i_1} \cdots T_{i_k}$ for $\mathbf{i} = (i_1 \dots i_k) \in Q_k$. Then Proposition 3.11 holds for the function ψ^s and we denote the corresponding number by $d_-(T_1, \dots, T_r)$. Under an additional assumption, one can show that this number is a lower bound for the Hausdorff dimension of Λ.

Theorem 4.1. *Let Λ be the limit set determined by affine contractions $S_i(x) = T_i x + a_i$ $(1 \le i \le r)$, so that $\Lambda = \bigcup_{i=1}^r S_i(\Lambda)$. If this union is disjoint, then* $\dim_H \Lambda \ge d_-(T_1, \dots, T_r)$.

The assumption that the sets $S_i(\Lambda)$ are pairwise disjoint cannot be replaced by the open set condition but the closed set condition surely suffices.

4.2. **Estimates under the bounded distortion condition.** In [11], Falconer estimated the box dimension of a mixing repeller of a non-conformal map, under the bounded distortion assumption. Let M be an open subset of $\mathbb{R}^d$, $f : M \to M$ a C^2 expanding map and Λ a repeller for f. We say that f has *bounded distortion* if

$$\|(d_x f)^{-1}\|^2 \|d_x f\| < 1 \quad \text{for all } x \in \Lambda.$$

This condition allows one to control the images of small balls under the iteration of f and in particular, to conclude that the images are convex sets. In [25], Manning and Simon constructed an example to show that this property does not hold without the bounded distortion condition.

Before addressing some results on dimension we recall some necessary facts from subadditive thermodynamic formalism in connection with mixing repellers of non-conformal maps.

A finite collection of closed sets $R_1, \cdots, R_r \subset \Lambda$ is called a *Markov partition* of the repeller Λ if

(1) $\Lambda = \bigcup_{i=1}^r R_i$ and $\overline{\mathrm{int} R_i} = R_i$;
(2) $\mathrm{int} R_i \cap \mathrm{int} R_j = \emptyset$ whenever $i \ne j$;
(3) $f(R_i) \supset R_j$ whenever $f(\mathrm{int} R_i) \cap \mathrm{int} R_j \ne \emptyset$.

It is well known that any repeller of an expanding map possesses Markov partitions of arbitrarily small size (see for example, [33]). Given a Markov partition $\{R_1, \cdots, R_r\}$ of Λ, we call a sequence $\mathbf{i} = (i_1 \cdots i_k) \in \{1, \cdots, r\}^k$ *admissible* if $f(R_{i_j}) \supset R_{i_{j+1}}$ for $j = 1, \dots, k-1$ (here k can be finite or ∞). Let Q_k be the collection of all admissible sequences of length k, in particular, $Q = Q_\infty$ is the associated topological Makov chain with the shift $\sigma : Q \to Q$ given by $\sigma(i_1 i_2 \dots) = (i_2 i_3 \dots)$. For $\mathbf{i} = (i_1 \cdots i_k) \in Q_k$, define the *cylinder*

$$R_{\mathbf{i}} = \bigcap_{j=1}^n f^{-j+1}(R_{i_j}).$$

We have the coding map $\chi : Q \to \Lambda$, defined by

$$\chi(i_1 i_2 \cdots) = \bigcap_{j=1}^{\infty} f^{-j+1}(R_{i_j}) = \bigcap_{n=1}^{\infty} R_{i_1 \cdots i_n}.$$

Now consider a *subadditive* sequence $\Phi = (\phi_n)_{n\in\mathbb{N}}$ of functions $\phi_n : \Lambda \to \mathbb{R}$, i.e., $\phi_{m+n}(x) \le \phi_n(x) + \phi_m(f^n x)$ for all $x \in \Lambda$, $n, m \in \mathbb{N}$. Furthermore, we assume that Φ satisfies the following conditions:

(1) $\left|\frac{1}{n}\phi_n(x)\right| \le a$ for some $a > 0$;
(2) $\{\phi_n\}$ has *bounded variation*, i.e., there exists $b > 0$ such that for all n,

$$\sup_{\mathbf{i}\in Q_n} \sup_{x,y\in R_{\mathbf{i}}} |\phi_n(x) - \phi_n(y)| \le b.$$

We define the topological pressure for the sequence of functions Φ by

$$\begin{aligned} P_\Lambda(f, \Phi) \quad &:= \quad \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{i}\in Q_n} \exp \sup_{x\in R_{\mathbf{i}}} \phi_n(x) \\ &= \quad \inf_n \frac{1}{n} \log \sum_{\mathbf{i}\in Q_n} \exp \sup_{x\in R_{\mathbf{i}}} \phi_n(x). \end{aligned}$$

One can show that $P_\Lambda(f, \Phi)$ is independent of the choice of the Markov partition.

For $n \in \mathbb{N}$ and $0 \le s \le d$, define

$$\phi_n^s(x) = \log \phi^s((d_x f^n)^{-1}), \tag{6}$$

where $\phi^s(\cdot)$ is the singular value function. It is easy to check that $\{\phi_n^s\}$ is subadditive, has bounded variations and $\left|\frac{1}{n}\phi_n^s(x)\right| \le a$ for some $a > 0$.

Theorem 4.2. *We have that*

$$\dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda \le s_0,$$

where s_0 is the unique root of the equation $P_\Lambda(f, \{\phi_n^s\}) = 0$.

The number s_0 can be thought of as a “natural” candidate for the dimension of Λ. Indeed, in [11], Falconer showed that $\dim_B \Lambda = s_0$ provided a certain topological requirement, called *Property P*, is satisfied. It prevents the drop in the value of the box dimension but holds only when $\dim_B \Lambda \ge d - 1$.

4.3. **Estimates via non-additive thermodynamic formalism.** In [3], Barreira obtained some sharp dimension estimates for repellers associated to $C^{1+\alpha}$ expanding maps. Unlike Falconer [11], he used a non-additive version of thermodynamic formalism. Later in [4], Barreira has established a more general dimension estimates for repellers associated to C^1-map without any additional assumptions.

We first recall some facts about the non-additive version of thermodynamic formalism (see [3], [30] for more details). Let $f : X \to X$ be a continuous map of a compact metric space and $\mathcal{U}$ a finite open cover of X. Given $U = (U_1, \dots, U_n) \in \mathcal{U}^n$ define the open set $X(U) = \bigcap_{k=1}^n f^{-k+1} U_k$ and the length $m(U) = n$. Consider a sequence $\Phi = (\phi_n)_{n\in\mathbb{N}}$ of functions $\phi_n : X \to \mathbb{R}$. For each $n \in \mathbb{N}$ let

$$\gamma_n(\Phi, \mathcal{U}) = \sup\{|\phi_n(x) - \phi_n(y)| : \ x, y \in X(U) \text{ for some } U \in \mathcal{U}^n\}$$

and denote by $|\mathcal{U}|$ the diameter of the cover $\mathcal{U}$. Here we assume that

$$\limsup_{|\mathcal{U}|\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n} = 0. \tag{7}$$

Given $U \in \mathcal{U}^n$, set $\Phi(U) = \sup_{X(U)} \phi_n$ if $X(U) \neq \emptyset$, and $\Phi(U) = -\infty$ otherwise. For each $Z \subset X$ and $\alpha \in \mathbb{R}$ let

$$M(Z, \alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha m(U) + \Phi(U)), \tag{8}$$

where the infimum is taken over all $\Gamma \subset \bigcup_{k\geq n} \mathcal{U}^k$ such that $\{X(U) : U \in \Gamma\}$ is a cover of Z. We also define

$$\underline{M}(Z, \alpha, \Phi, \mathcal{U}) = \liminf_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha n + \Phi(U)), \tag{9}$$

$$\overline{M}(Z, \alpha, \Phi, \mathcal{U}) = \limsup_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha n + \Phi(U)), \tag{10}$$

where the infimum is taken over all $\Gamma \subset \mathcal{U}^n$ such that $\{X(U) : U \in \Gamma\}$ is a cover of Z. One can show that when α runs from $-\infty$ to ∞ each of the quantities in (8), (9) and (10) jumps from ∞ to 0 at a unique critical value. Hence define

$$\begin{aligned} P_Z(\Phi, \mathcal{U}) &= \inf\{\alpha \in \mathbb{R} : M(Z, \alpha, \Phi, \mathcal{U}) = 0\}, \\ \underline{CP}_Z(\Phi, \mathcal{U}) &= \inf\{\alpha \in \mathbb{R} : \underline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\}, \\ \overline{CP}_Z(\Phi, \mathcal{U}) &= \inf\{\alpha \in \mathbb{R} : \overline{M}(Z, \alpha, \Phi, \mathcal{U}) = 0\}. \end{aligned}$$

One can show that the limits

$$P_Z(\Phi) = \lim_{|\mathcal{U}|\to 0} P_Z(\Phi, \mathcal{U}),$$

$$\underline{CP}_Z(\Phi) = \lim_{|\mathcal{U}|\to 0} \underline{CP}_Z(\Phi, \mathcal{U}),$$

$$\overline{CP}_Z(\Phi) = \lim_{|\mathcal{U}|\to 0} \overline{CP}_Z(\Phi, \mathcal{U})$$

exist. The numbers $P_Z(\Phi)$, $\underline{CP}_Z(\Phi)$ and $\overline{CP}_Z(\Phi)$ are called respectively the *non-additive topological pressure* and *non-additive lower* and *upper capacity topological pressures* of Φ on the set Z with respect to f. Note that the definition of the non-additive topological pressure coincides with the classical definitions of topological pressure in the additive or subadditive cases.

Let $\Phi^s = (\phi_n^s)_{n\in\mathbb{N}}$ be a sequence of functions $\phi_n^s : X \to \mathbb{R}$ for each s in some interval $I \subset \mathbb{R}$. We assume that

(1) Φ^s satisfies (7) for each $s \in I$;
(2) there exists $c_1 < 0$ and $c_2 < 0$ such that if $x \in X$, $n \in \mathbb{N}$ and $s, t \in I$ with $s \neq t$, then $c_1 n \leq (\phi_n^s(x) - \phi_n^t(x))/(s-t) \leq c_2 n$.

Proposition 4.3. *If a sequence of function Φ^s satisfies the above assumptions then*

(1) *the functions $s \mapsto P_Z(\Phi^s)$, $s \mapsto \underline{CP}_Z(\Phi^s)$ and $s \mapsto \overline{CP}_Z(\Phi^s)$ are strictly decreasing and Lipschitz continuous;*
(2) *there exists unique numbers $s_P \leq s_{\underline{CP}} \leq s_{\overline{CP}}$ such that*

$$P_Z(\Phi^{s_P}) = \underline{CP}_Z(\Phi^{s_{\underline{CP}}}) = \overline{CP}_Z(\Phi^{s_{\overline{CP}}}) = 0.$$

4.3.1. *The case of expanding $C^{1+\alpha}$-maps.* In [3], Barreira considered a repeller Λ for an expanding $C^{1+\alpha}$-map f and gave some effective dimension estimates.

Some simple but rather rough estimates can be obtained by taking two Hölder continuous functions $\underline{\phi}$ and $\overline{\phi}$ on Λ given by

$$\underline{\phi}(x) = -\log \|d_x f\|, \quad \overline{\phi}(x) = \log \|(d_x f)^{-1}\|.$$

Let $\underline{t}$ and $\overline{t}$ be the unique roots of Bowen's equations

$$P_\Lambda(t\underline{\phi}) = 0, \quad P_\Lambda(t\overline{\phi}) = 0.$$

Theorem 4.4. *The following inequalities hold:*

$$\underline{t} \leq \dim_H \Lambda \leq \underline{\dim}_B \Lambda \leq \overline{\dim}_B \Lambda \leq \overline{t}. \tag{11}$$

Using non-additive thermodynamic formalism one can obtain rather sharper dimension estimates under certain additional assumptions. Namely, given $\delta \in (0, 1]$, we say that the map f is δ-bunched if for every $x \in \Lambda$,

$$\|(d_x f)^{-1}\|^{1+\delta} \|d_x f\| < 1.$$

Note that the map f satisfying the bounded distortion property is 1-bunched. Consider two sequences of functions on Λ defined by

$$\underline{\Phi} = \{\underline{\phi}_n(x) = -\log \|d_x f^n\|\}, \quad \overline{\Phi} = \{\overline{\phi}_n(x) = \log \|(d_x f^n)^{-1}\|\}.$$

Theorem 4.5. *Let f be a $C^{1+\alpha}$ expanding α-bunched map. Then*

$$\underline{s} \le \dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda \le \overline{s}, \tag{12}$$

where $\underline{s}$ and $\overline{s}$ are the unique roots of Bowen's equations

$$\overline{CP}_\Lambda(s\underline{\Phi}) = 0, \quad P_\Lambda(s\overline{\Phi}) = 0.$$

Note that the lower and upper estimates in (11) and (12) cannot be improved. If f is a α-bunched $C^{1+\alpha}$ expanding map, then

$$\underline{t} \le \underline{s} \le \overline{s} \le \overline{t}.$$

If f is conformal, then $\underline{t} = \underline{s} = \overline{s} = \overline{t}$. If f is not conformal, the numbers $\underline{s}$ and $\overline{s}$ provide sharper estimates than the numbers $\underline{t}$ and $\overline{t}$, see [3].

4.3.2. *The case of expanding C^1-maps.* In [4], Barreira, following [11], established upper dimension estimates for repellers of expanding C^1-map without using the bounded distortion or the δ-bunched conditions.

Let $f : M \to M$ be a C^1 expanding map of a smooth manifold and Λ an invariant repeller associated to f. Set $m = \dim M$, and choose $\varepsilon > 0$ such that f is invertible on each ball $B(x, \varepsilon)$ with $x \in \Lambda$. Given $s \in [0, m]$, we define a sequence Φ^s of functions $\phi^s_n : \Lambda \to \mathbb{R}$ by

$$\phi^s_n(x) = \log \sup\{\phi^s((d_y f^n)^{-1}) : \ y \in B_n(x, \varepsilon)\},$$

where $B_n(x, \varepsilon) = \bigcap_{k=0}^{n-1} f^{-k} B(f^k x, \varepsilon)$ and $\phi^s(\cdot)$ is the singular value function. By Proposition 4.3, there exists a unique number s such that $P_\Lambda(\Phi^s) = 0$.

Theorem 4.6. *Let Λ be a repeller for a C^1 expanding map. Then $\overline{\dim}_B \Lambda \le s$.*

One can obtain even better estimate using Markov partitions $\{R_1, \cdots, R_r\}$ of Λ. Let (Q, σ) be the associated Markov chain and $\chi : Q \to \Lambda$ the coding map, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$. Now define the set

$$R_{x,n} := \bigcup_{\chi(i_1 i_2 \cdots) = x} \bigcap_{j=1}^{n} f^{-j+1} R_{i_j}.$$

Given $s \in [0, m]$, define a sequence $\widetilde{\Phi}^s$ of functions $\widetilde{\phi}^s_n : \Lambda \to \mathbb{R}$ by

$$\widetilde{\phi}^s_n(x) = \log \sup\{\phi^s((d_y f^n)^{-1}) : \ y \in R_{x,n}\}.$$

There exists a unique number $\tilde{s}$ such that $P_\Lambda(\widetilde{\Phi}^{\tilde{s}}) = 0$.

Theorem 4.7. *Let Λ be a repeller for a C^1 expanding map. Then $\overline{\dim}_B \Lambda \le \tilde{s}$.*

Note that one can choose a Markov partition of diameter at most $\varepsilon/2$. This implies that $R_{x,n} \subset B_n(x,\varepsilon)$ for each $x \in \Lambda$ and $n \in \mathbb{N}$. Hence, $\widetilde{\Phi}^s \le \Phi^s$ and $\tilde{s} \le s$.

Let us compare these results with the result by Falconer in [11]. Namely, Falconer considered the sequence of functions Φ^s_* (see (6)). Let s_* be the unique root of Bowen's equation $P_\Lambda(\Phi^s_*) = 0$. In the case when f is an expanding C^2-map satisfying the bounded distortion property, s_* gives us an upper bound for $\overline{\dim}_B\Lambda$. For a general C^1 map f the number s_* may no longer provide an upper bound for the upper box dimension. In [38], a different type of pressure functional $P'_\Lambda(\cdot)$ was introduced, and it was shown that the number s'_* – the unique root of Bowen's equation $P'_\Lambda(\Phi^s_*) = 0$ – provides an upper bound for $\dim_H\Lambda$. In [1], the authors proved that $P'_\Lambda(\cdot)$ is equivalent to $P_\Lambda(\cdot)$. Hence $s_* = s'_*$. Therefore, s_* gives an upper bound for the Hausdorff dimension.

On the other hand, one can see that $\Phi^s_* \le \widetilde{\Phi}^s \le \Phi^s$ for every s and hence $s_* \le \tilde{s} \le s$. Barreira [4] showed that if f is a $C^{1+\alpha}$ map, which is α-bunched on Λ for some $\alpha > 0$, then $s_* = \tilde{s} = s$. Therefore by Theorem 4.7, $\overline{\dim}_B\Lambda \le s_* = \tilde{s} = s$. The particular case when $\alpha = 1$ is exactly the one considered by Falconer in [11].

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