1 Introduction

We recall that $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$, with operations $\oplus = \text{max}$, $\otimes = +$, and that $\mathbb{R}_{\text{max}}^n = \mathbb{R}_{\text{max}} \times \cdots \times \mathbb{R}_{\text{max}}$ ($n$ times) with the operations $X \oplus Y = (x_1 \oplus y_1, \ldots, x_n \oplus y_n)$ and $\alpha \otimes X = \alpha X = (\alpha x_1, \ldots, \alpha x_n)$. A line segment in max-plus between $X, Y \in \mathbb{R}_{\text{max}}^n$ would be,

$$ [X, Y] = \{ \alpha X \oplus \beta Y | \alpha \oplus \beta = \epsilon \} = \{ \text{max}(\alpha + X, \beta + Y) | \text{max}(\alpha, \beta) = 0 \}. \quad (1) $$

We recall from [1] that a line segment in $\mathbb{R}_{\text{max}}^2$, and consequently $\mathbb{R}_{\text{max}}^n$ has a similar case, has one of the three following forms:

When $X \leq Y$ and $x_1 - y_1 \leq x_2 - y_2$, then

$$ [X, Y] = [X, (y_1 + x_2 - y_2, x_2)] \cup [(y_1 + x_2 - y_2, x_2), Y] \quad (2) $$

When $X \leq Y$ and $x_2 - y_2 \leq x_1 - y_1$, then

$$ [X, Y] = [X, (x_1, y_2 + x_1 - y_1)] \cup [(x_1, y_2 + x_1 - y_1), Y] \quad (3) $$

When $X \not\leq Y$ and $Y \not\leq X$, then

$$ [X, Y] = [X, \text{max}(X, Y)] \cup [\text{max}(X, Y), Y] \quad (4) $$
We notice that the first possible line segment is given in the first quadrant, the second possible line segment is in the second quadrant, and the third line segment is in the third segment. The fourth quadrant contains the line segments where \( x_1 = y_1, \ x_2 = y_2, \) and \( y_1 - x_1 = y_2 - x_2. \)

A subset \( S \subseteq \mathbb{R}^n_{\text{max}} \) is said to be convex if \([x, y] \subseteq S\) for all \( x, y \in S\). We recall from [1], that \( S \subset \mathbb{R}^n_{\text{max}} \) is a semispace at \( z \in \mathbb{R}^n_{\text{max}} \) if \( S \) is a maximal convex subset of \( \mathbb{R}^n_{\text{max}} \) avoiding \( z \). For the set \( S \) to be a maximal convex set, then there exist no convex set \( S' \neq S \), where \( S \subset S' \) and \( z \notin S' \). Note that for finite points we can take \( z \) to be the point \((0,0,...,0)\), since a semispace at any other point results from a translation of the semispaces around the origin. Nitica and Singer [1] determined that there are exactly \( n + 1 \) semispaces at \( z \), namely

\[
S_0 = \{ X \in \mathbb{R}^n_{\text{max}} | 0 < \max(x_1, \ldots, x_n) \} = \{ X \in \mathbb{R}^n_{\text{max}} | 0 < x_1 \} \cup \ldots \cup \{ X \in \mathbb{R}^n_{\text{max}} | 0 < x_n \} = \{ X \in \mathbb{R}^n_{\text{max}} | X \not< 0 \}
\]

(5)

\[
S_k = \{ X \in \mathbb{R}^n_{\text{max}} | x_k < \max(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, 0) \}
\]

\[
= \{ X \in \mathbb{R}^n_{\text{max}} | x_k < 0 \} \cup \ldots \cup \{ X \in \mathbb{R}^n_{\text{max}} | x_k < x_{k-1} \} \cup \{ X \in \mathbb{R}^n_{\text{max}} | x_k < x_{k+1} \} \cup \ldots \cup \{ X \in \mathbb{R}^n_{\text{max}} | x_k < x_n \}.
\]

(6)

In this paper we determine the hemispaces in \( \mathbb{R}^n_{\text{max}} \). To do so, we considered the complements of the semispaces centered at the origin. By [1] these complements are given by

\[
\mathbb{C}S_0(z) = \{ X \in \mathbb{R}^n_{\text{max}} | 0 \geq \max(x_1, \ldots, x_n) \}
\]

(7)

\[
\mathbb{C}S_k(z) = \{ X \in \mathbb{R}^n_{\text{max}} | x_k \geq \max(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n, 0) \}.
\]

(8)

Since these complements intersect one another at their boundary points, we consider them without boundaries, and then determined how we could partition \( \mathbb{R}^n_{\text{max}} \) into two convex subsets made up of semispace complements and their boundaries.
Definition 1.1 A set $H \subseteq \mathbb{R}^n_{\text{max}}$ is a hemispace in $\mathbb{R}^n_{\text{max}}$ if both $H$ and $\complement H$ are convex.

Let $S_i$ be a semispace and $\complement S_i$ be the complement of that semispace where $x_i$ is the greatest variable in the space and $x_0 = 0$. For notational ease, we refer to $\complement S_i/\text{bd} \complement S_i$ in the rest of the paper by $\complement S_i$.

Definition 1.2 A face of dimension $k$ will be referred to as a $k$-face for the remainder of the paper.

Definition 1.3 A hyperplane of the form:

$$x_0 \oplus \ldots \oplus x_n = x_m \oplus \ldots \oplus x_n; m \leq n$$

(9)

has a boundary hyperplane of:

$$x_0 \oplus \ldots \oplus x_{m-1} = x_m \oplus \ldots \oplus x_n$$

And has the section where $x_m, \ldots, x_n > x_0, \ldots, x_{m-1}$ filled.

2 Hemispaces in $\mathbb{R}^2_{\text{max}}$

We note that $\complement S_i$ is convex with or without the boundary. We also know that semispaces are the building blocks of hemispaces.

Lemma 2.1 $\complement S_i/\text{bd} \complement S_i$ is a minimal non-trivial hemispace.

Proof Let $H$ be a minimal non-trivial hemispace. We first consider $H$ to be a single point, $H = \{P\}; P = (p_1, \ldots, p_n)$. However, $P$ is not a hemispace since $\mathbb{R}^n_{\text{max}}/\{P\}$ is not convex because there exist points $Q = (q_1, p_2, \ldots, p_n)$ and $R = (r_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n_{\text{max}}/\{P\}$ such that $q_1 < p_1 < r_1$, so $P \in [Q, R]$. Thus, we consider adding a point $Q$ to $H$. To maintain convexity in $H$, we must have $[P, Q] \subseteq H$ as well. If $[P, Q]$ is not degenerate (has 2 connecting segments for the case $n = 2$), then clearly both of those segments must be made half-lines to maintain the convexity of $\complement H$. Otherwise there exist $S, T \in \complement H$ located on the full line containing one of those segments, clearly a non-empty $A \subseteq [P, Q]$ also has $A \subseteq [S, T]$. Now we have two half lines that start at,

$$(q_1 + p_2 - q_2, p_2) \text{ if } Q < P$$

$$(p_1, q_2 + p_1 - q_1) \text{ if } P < Q$$

$$\max(P, Q) \text{ if } P \not\in Q, Q \not\in P$$

and extend through $P$ and $Q$. Note that this creates the boundary of a semispace. Clearly, to maintain convexity, $H$ must also contain the entire space either above or below these half lines. One side gives a semispace, and the other gives its complement. Clearly the complement is smaller, which we can make even smaller by excluding the boundary.

If the segment connecting $P$ and $Q$ is degenerate, then for the case $n = 2$, either $q_1 = p_1$ or $q_2 = p_2$. Without loss of generality, suppose $q_2 = p_2$ and $q_1 > p_1$, so the line line connecting them is a horizontal line. Clearly we must extend this segment to a half-line and include the space either above or below it. If we extend it to the left and include the space below it, we get $\complement S_0$. If we extend left and include the space above the line, then it is clear we must also include half-line starting at $Q$ parallel to the main bisector to maintain the convexity of $\complement H$, and we end up with $\complement S_2$. Finally if we extend to the right, and include below, we must
also have the half line starting at $P$, parallel to the main bisector, and we get $\mathcal{C}S_{1}$. If we include above, it is clear that $\mathcal{U}H$ is not convex. The only way to fix this problem is to include the other vertical (or horizontal) half-line starting at $P$ and the section to the right (or above) it. This is the only hemispace that can be formed without being the union of semispace complements, but it is clear that these hemispace are larger then any single semispace complement.

**Remark 2.2** Because of this previous lemma, it makes sense to look at the semispaces, and hence there complements, when trying to determine how a hemispace would work. Let us look at $R$ because of this previous lemma, it makes sense to look at the semispaces, and hence there complements, when trying to determine how a hemispace would work. Let us look at $R$ then any single semispace complement.

$$\mathcal{C}S_{0} = \{X \in \mathbb{R}_{\text{max}}^{2} | 0 \geq \max(x_{1}, x_{2})\}$$

(10)

$$\mathcal{C}S_{1} = \{X \in \mathbb{R}_{\text{max}}^{2} | x_{1} \geq \max(0, x_{2})\}$$

(11)

$$\mathcal{C}S_{2} = \{X \in \mathbb{R}_{\text{max}}^{2} | x_{2} \geq \max(0, x_{1})\}$$

(12)

and their boundaries to form these hemispaces. We notice that $\mathcal{C}S_{0}$ is the negative quadrant with the boundry of $\mathcal{C}S_{0}$ being the $1 - faces$ $(x_{1}, 0); x_{1} < 0$ and $(0, x_{2}); x_{2} < 0$ along with the origin.

**Example 2.3** The first basic hemispaces in $\mathbb{R}_{\text{max}}^{2}$ that we come across is $S_{0}$ and $\overline{C}S_{0}$. What we want to see is how we can split and distribute the boundary between them such that both $S_{0}$ and $\overline{C}S_{0}$ remain convex and by that, hemispaces.

Let us define $A = S_{0}$ and $B = \overline{C}S_{0}$ without the boundary. $l_{1} = (x_{1}, 0); x_{1} < 0$, $l_{2} = (0, x_{2}); x_{2} < 0$, and $0 = (0, 0)$. We immediately have two basic cases were $l_{1}, l_{2}, 0 \in A$ or $l_{1}, l_{2}, 0 \in B$.

Now we can see that if both $l_{1}, l_{2} \in A$ then for $A$ to stay convex $[l_{1}, l_{2}] \in A$

$$[l_{1}, l_{2}] = [(x_{1}, 0), (0, x_{2})] = \{(\max(\alpha + x_{1}, \beta), \max(\alpha, \beta + x_{2})\}$$

since $x_{1}, x_{2}, \alpha, \beta \leq 0$ then $\{(\max(\alpha + x_{1}, \beta), \max(\alpha, \beta + x_{2})\} = (0, 0)$ which implies that $0 \in [l_{1}, l_{2}] \in A$. This is symmetric to $[l_{1}, l_{2}] \in B$.

**Lemma 2.4** *If $l_{1}, l_{2} \in \mathbb{R}_{\text{max}}^{2}$ are in the same convex space, then 0 is in that convex space.*

**Proof** The proof of Lemma 2.4 follows directly from our previous example.

Let us now consider the case were $l_{1}$ and $l_{2}$ are not in the same convex space. Let $l_{1} \in A$. Then consider the line segment $[(-2, 0), (2, 0)]$ with $(-2, 0) \in l_{1}$ and $(2, 0) \in A$.

$$[(-2, 0), (2, 0)] = \{(\max(\alpha - 2, \beta + 2), \alpha, \beta)\}$$

if we let $\alpha = 0$ and $\beta = -2$, then $\{(\max(\alpha - 2, \beta + 2), \alpha, \beta)\} = \{(0, 0)\}$. Which implies that $0 \in [(-2, 0), (2, 0)] \in A$. Similarly, if $l_{2} \in A$ then $0 \in A$.

**Remark 2.5** Notice that this result say’s that if either $l_{1}$ or $l_{2} \in A$ then $0 \in A$. So flipping these results we notice that if $0 \in B$, then neither $l_{1}$ or $l_{2}$ can be in $A$, therefore they both must be in $B$. Giving us a trivial result with the entire boundary in $B$.

Because of this result and Lemma 2.4 all possible splits of the boundary in $\mathbb{R}_{\text{max}}^{2}$ for $S_{0}$ are:

$l_{1}, l_{2}, 0 \in A$

$l_{1}, l_{2}, 0 \in B$
\[ l_1, 0 \in A \text{ and } l_2 \in B \]
\[ l_2, 0 \in A \text{ and } l_1 \in B \]

We have a similar case when considering \( S_1, \mathcal{C}S_1 \) and \( S_2, \mathcal{C}S_2 \).

**Remark 2.6** It is easy to see that some of these properties for \( R_{\text{max}}^2 \) could be generalized to work for \( R_{\text{max}}^n \). But as you may notice for the case were \( n = 2 \), since we only have three semispaces complements, we are always only seperating one of those complements from the rest. In section 3 we continue to work on the one section split in \( R_{\text{max}}^n \) and in section 4 we look at what will happen if we split more then one section apart, which can not happen unless \( n \geq 3 \).

### 3 Spliting the Boundary of one Sector in \( \mathbb{R}_{\text{max}}^n \)

Recall from [2] that the boundary of \( \mathcal{C}S_i \) is a hyperplane. In this section we consider how to split one of these hyperplanes between \( S_i \) and \( \mathcal{C}S_i \) such that we have two hemispaces. Without loss of generality let us look at \( S_0 \) and \( \mathcal{C}S_0 \). We know from the last section that for \( n = 2 \) if we have the origin in \( \mathcal{C}S_0 \) then all the connecting one dimensional lines are in \( \mathcal{C}S_0 \). We can actually generalize this result to \( R_{\text{max}}^n \). First let us make neccesary definitions.

**Definition 3.1** \( X = (x_1, x_2, ..., x_n) \) is a \( m \) face with \( m < n \) if there exist exactly \( m \) unique coordinates in \( X \), all non-unique coordinates either are equal to zero or the same \( x_i \) where \( x_i \) is one of the unique coordinates.

**Remark 3.2** If the non-unique coordinates in \( X \) are all dependent on a unique \( x_i \) then this \( m \) face is connected to the one dimensional main bisector. If the non-unique coordinates in \( X \) are equal to 0, then the \( m \) face is connected to the negative octant.

**Definition 3.3** An \( m+1 \) face \( Y \) is said to be adjacent to \( X \) iff the same \( m \) unique coordinates in \( X \) are also unique in \( Y \), and \( Y \) has one additional unique coordinate.

**Example 3.4** Let \( X = (x_1, x_2, x_3, 0, 0, 0) \) and let \( Y = (y_1, y_2, y_3, 0, y_5, 0) \). \( X \), a \( 3 \) face, and \( Y \), a \( 4 \) face, are adjacent because in both \( X \) and \( Y \), \( x_i \) and \( y_i \) are independant coordinates for \( i \in [1, 2, 3] \), while \( x_5 \) is 0 in \( X \) and \( y_5 \) is an independant coordinate in \( Y \).

**Remark 3.5** An \( m \) face \( X \) can not be adjacent to a \( m+k \) face \( Y \), where \( |k| \geq 2 \).

**Lemma 3.6** If a \( m \) face with \( m < n \) is a part of the boundary of \( \mathcal{C}S_i \), then all adjacent \( m+1 \) faces are in \( \mathcal{C}S_i \).
**Proof** Let $X$ be an $m - \text{face}$ with $m < n$, and $X \in \mathbb{CS}_i$. Let $Y$ be a $m + 1 - \text{face}$ and adjacent to $X$. Assume $Y \in \mathbb{S}_i$

We want to show that $\exists Z \in \mathbb{S}_i$ s.t. $X \in [Y, Z]$

Let $X = (x_1, x_2, ..., x_m, 0, ..., 0)$. Then, $Y = (x_{1k}, x_{2k}, ..., x_{mk}, y_1, y_2, ..., y_k)$, where $m + k = n$, and $\exists y_i \in \mathbb{R}_{\max}, i \in [1, 2, ..., k], y_i \neq 0$, and $y_j = 0, j \neq i$ and $j \in [1, 2, ..., k]$. Let $Z = (x_{1k}, x_{2k}, ..., x_{mk}, -y_1, -y_2, ..., -y_k)$. Where $-y_i$ is chosen s.t. $-y_i$ is the same sign as $y_i$, and $Z \in \mathbb{S}_i$.

$[Y, Z] = \{(x_{1k}, x_{2k}, ..., x_{mk}, y_1, y_2, ..., y_k), (x_{1k}, x_{2k}, ..., x_{mk}, -y_1, -y_2, ..., -y_k)\}$

$\{(\max(\alpha + x_{1k}, \beta + x_{2k}), \max(\alpha + x_{mk}, \beta + x_{mk}), \max(\alpha + y_1, \beta - y_1), ..., \max(\alpha + y_k, \beta - y_k))\}$

max($\alpha + x_{ik}, \beta + x_{ik}) = x_{ik}; i \in [1, 2, ..., m]$

When $y_j = 0$, then max($\alpha + y_j, \beta - y_j) = 0; j \in [1, ..., k]$

When $y_j \neq 0$, then $\max(\alpha + y_j, \beta - y_j) = 0$, when $\alpha = 0$ and $\beta = y_j$ if $y_j < 0$ or when $\alpha = -y_j$ and $\beta = 0$ is $y_j > 0$.

Then $X \in [Y, Z]$.

This result immediately gives a corollary about a one sector split in $\mathbb{R}_{\max}^n$.

**Corollary 3.7** If the center is part of the boundary containing only $\mathbb{CS}_i$, then we have the trivial hemispace were the entire boundary is containing only $\mathbb{CS}_i$

**Remark 3.8** Corollary 3.7 tells us that all non-trivial hemispaces for a one sector split of the boundary is going have to have the origin part of the boundary of $\mathbb{S}_i$. We can recall from Lemma 2.2, that no more the one line of dimension one can be in the same convex section without the origin for $n = 2$. This brings us to are next lemma and a generalization of this result for $\mathbb{R}_{\max}^n$.

**Lemma 3.9** If the center and all faces up to dimension $m < n$ are in $\mathbb{S}_i$, then no more then one $m+1-\text{face}$ is in $\mathbb{CS}_i$.

**Proof** Let the center and all faces up to dimension $m < n$ be in $\mathbb{S}_i$. Let $T$ be the number of $m + 1 - \text{faces}$ that are in $\mathbb{CS}_i$. Assume $T \geq 2$. Since $T \geq 2$ then we can choose two faces of degree $m + 1$, $E_1$ and $E_2 \in \mathbb{CS}_i$. Let $E_1 = \{(x_1, x_2, ..., x_n)|x_i \neq 0$ for exactly $m + 1$ $i$'s, and $x_1 = x_i$ or $x_i = 0$ for the remaining $n - (m + 1)$ points$\}$ and $E_2 = \{(y_1, y_2, ..., y_n)|y_i \neq 0$ for exactly $m + 1$ $i$'s, and $y_j = y_i$ or $y_j = 0$ for the remaining $n - (m + 1)$ points$\}$ and $E_1 \neq E_2$.

$[E_1, E_2] = [(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)] = \{(\max(\alpha + x_1, \beta + y_1), ..., \max(\alpha + x_n, \beta + y_n))\}$

Case 1: $E_1 < E_2$ similarly $E_2 < E_1$.

Case 2: $0 < E_1 < E_2$ similarly $0 < E_2 < E_1$.

Case 3: $E_1 \not< E_2$ and $E_2 \not< E_1$

Case 1: Let $\alpha = 0$ and $\beta = -\max(y_i|i \in [1, 2, ..., n])$ then $[E_1, E_2] = \{(\max(x_1, y_1 - y_i), ..., \max(x_n, y_n - y_i))\}$. This has at most $m$ unique points since the $i^{th}$ point was changed to 0. Then $\exists k \in K a p - face$ with $p < m + 1$ s.t. $k \in [E_1, E_2]$. Therefore, both $E_1$ and $E_2$ cannot be in $\mathbb{CS}_i$.

Case 2: Let $0 \leq x_n \leq ... \leq x_1 \leq y_1 \leq ... \leq y_n$ and let $\alpha = 0$.

$[E_1, E_2] = \{(\max(x_1, \beta + y_1), ..., \max(x_n, \beta + y_n))\}$. Since $y_n > x_1 > 0$, then $\exists y_0 s.t. y_0 + \beta_0 = x_1$ and $\max(x_1, \beta + y_1) = x_1; y_1 < y_0$. Then, $[E_1, E_2] = \{(x_1, x_2, ..., x_1, x_1, ..., x_1)\}$. Since we made some independent variables dependent on $x_1$, then $\exists k \in K a p - face$ with $p < m + 1$ s.t. $k \in [E_1, E_2]$. Therefore, both
Case 3: Since $E_1 \not\subseteq E_2$ and $E_2 \not\subseteq E_1$, then $[E_1, E_2] = [E_1, \max(E_1, E_2)] \cup [\max(E_1, E_2), E_2]$. Let $x_i > y_i; i \in [1, ..., r]$ and $y_j > x_j; j \in [r + 1, ..., n]$. Then, $[E_1, \max(E_1, E_2)] = [(x_1, ..., x_n), (x_1, ..., x_r, y_{r+1}, ..., y_n)] = \{ (x_1, ..., x_r, \max(\alpha + x_{r+1}, \beta + y_{r+1}), ..., \max(\alpha + x_n, \beta + y_n)) \}$. Let $r < m + 1$, if $r \geq m + 1$ look at the line segment $[\max(E_1, E_2), E_2]$. Rest of the proof is similar to case 2.

Therefore $T < 2$.

**Remark 3.10** With Lemma 3.6 and Lemma 3.9 we can determine all boundary splits between $S_i$ and $\mathcal{C}S_i$, both trivial and non-trivial, for $\mathbb{R}_\text{max}^n$ by first choosing where our origin lies, and then using the lemma’s to determine where higher dimensional faces must fall. Once we stumble upon a face that can go on either face, we choose were we want this face to fall, and once again the Lemma’s show that more higher dimensional faces will follow. We continue this process until all faces are accounted for. We go over all of the combinatorics of this later in this paper.

### 4 Multi-Sector Split in $\mathbb{R}_\text{max}^n$

Since $\mathbb{R}_\text{max}^n$ has $n + 1$ sectors as mentioned earlier. The first $n$ that can have a multi-sector split would be $n = 3$. For looking at how these splits can happen we use two lemma’s, without proof, for a general idea. We will later show and algebraic result with proof that works for all types of splits in all dimensions.

**Lemma 4.1** Let a face have $k$ sectors. Then if all $k - 1$ faces are bordering that section, then the whole boundary borders that section.

**Lemma 4.2** If all faces up to dimension $m < n$ are bordering the same section, then no more than one $m + 1$ face can border the other section.

**Remark 4.3** Lemma 4.1 and Lemma 4.2 give general rules to follow when splitting the boundary of a $k$ sector split but this leaves a lot of choices undetermined. For these reason we took a different approach to solving this problem using some set theory and combinatorics. We begin with a lemma showing that the boundary must stay connected and a line segment on the boundary stay’s on the boundry, while also introducing some new notation, which will lead us to our main theorem.

**Lemma 4.4** Let $H \subseteq \mathbb{R}_\text{max}^n$ be a hemispace and $S \subseteq \mathbb{R}_\text{max}^n$ be a $k$ face. If there exists $p \in S$ such that $p \in H$, then $S \subseteq H$.

**Proof** Observe that it suffices to show that for a given $q \in S$, either for each $\mathcal{C}S_i$ there exists $r_i \in \mathcal{C}S_i$ such that $p \in [q, r_i]$ or for each $\mathcal{C}S_i$ there exists $s_i \in \mathcal{C}S_i$ such that $q \in [p, s_i]$ since this would establish that $p$ and $q$ must be in the same hemispace. Furthermore, we must only find such $r_i$ or $s_i$ for each $\mathcal{C}S_i$ that is bounded by $S$ because if all such $\mathcal{C}S_i$ are in $H$ (or all not in $H$), then $S$ is not a boundary of $H$. We will do this
by showing that each point in the lines that go through $p$ and are parallel to $1 - faces$ (the axes or main bisector) that bound $S$ must be in $S$. This process will be independent of $p$, and so once we know that a half-line going through $p$ and parallel to one $1 - face$ is in $H$, we can then repeat the the process for a line parallel to a different $1 - face$ for each point on that half line, thus tracing out a $2 - face$ cross section of the $k - face$, and so on until we trace out the entire $k - face$ with half-lines parallel to the $k 1 - faces$ that $S$ is bounded by.

First, suppose $S$ is of the form $x_{k+1} = x_{k+2} = \ldots = x_n > 0, x_1, \ldots, x_k$ so $p = (p_1, \ldots, p_k, p_0, \ldots, p_0)$ where $p_0 > p_1, \ldots, p_k$. To show that every point in $S$ on the half-line that goes through $p$ and is parallel to the $x_i$ axis must also be in $H$, consider a given point $q \neq p$ on that half-line, so $q = (p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_k, p_0, \ldots, p_0)$ where $p_0 > p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_k$. Observe that since $p$ and $q$ only differ by one coordinate, we must have either $p > q$ or $p < q$. Without loss of generality, assume $q > p$ (i.e. $q_i > p_i$); the other case follows from interchanging $p$ and $q$. For each $j \in \{k+1, \ldots, n\}$ choose $r_j = (p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_k, 0, \ldots, 0, p_0, 0, \ldots, 0)$ where the $p_0$ is the $j$th coordinate so that $r_j \in \mathbb{CS}_j$. Thus, $\max(r_j, p) = q$, so $q \in [r_j, p]$.

Now, consider a given $q$ on the half-line through $p$ that is parallel to the main bisector, so $q = (p_1, \ldots, p_k + c, p_0 + c, \ldots, p_0 + c)$. Without loss of generality, assume $q > p$, i.e. $c > 0$. Then for each $j \in \{k+1, \ldots, n\}$ choose $r_j = (p_1, \ldots, p_k, 0, \ldots, 0)$ where $p_0$ is the $j$th coordinate so that $r_j \in \mathbb{CS}_j$. Then $\max(\alpha r_j, \beta q) = p$ for $\alpha = 0$ and $\beta = -c$, so $p \in [r_j, q]$. Now, suppose $S$ is of the form $0 = x_{k+1} = \ldots = x_n > 0, x_1, \ldots, x_k$, so $p = (p_1, \ldots, p_k, 0, \ldots, 0)$ where $0 > p_1, \ldots, p_k$. Let $q \neq p$ be a point on the half-line passing through $p$ and parallel to the $x_i$ axis, so $q = (p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_k, 0, \ldots, 0)$. Without loss of generality, assume $q > p$, i.e. $q_i > p_i$. For each $j \in \{k+1, \ldots, n\}$ choose $r_j = (p_1, \ldots, p_{i-1}, q_i + 1, p_{i+1}, \ldots, p_k, 0, 0, \ldots, 0, 0, \ldots, 0)$ where the $l$th coordinate $1$ is the $j$th coordinate so that $r_j \in \mathbb{CS}_j$. Thus, $\max(\alpha r_j, \beta q) = q$ for $\alpha = -1$ and $\beta = 0$, so $q \in [r_j, p]$. Finally, for $\mathbb{CS}_0$, choose $r_0 = (p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_k, -1, \ldots, -1) \in \mathbb{CS}_0$. Then $\max(r_0, p) = q$, so $q \in [r_0, p]$.

**Remark 4.5** The basic principle that this proof is saying is that if any general point of a $k - face$ is part of a hemisphere, then the entire $k - face$ is part of that hemisphere. This also means that you can not split a $k - face$ between two hemispheres. This fact has been taken for granted up until now, but it is important to prove for the next theorem.

**Lemma 4.6** Suppose $S_1, S_2$ are faces in $\mathbb{R}^n_{\max}$ of dimension $K$ and $L$ respectively, bordering the same hemisphere, with $K \leq l \leq n$. Suppose the highest dimensional boundary of $S_1$ and $S_2$ is $S_3$ with dimension $m \leq k$. Then for $p \in S_1$ and $q \in S_2$, the segment $[p, q] \subseteq S_1 \cup S_2 \cup S_3$ and $[p, q] \cap S_1 \neq \emptyset; \ i \in \{1, 2, 3\}$.

**Proof** For notational purposes, consider 0 to be another variable $x_i$, so that there are a total of $n + 1$ variables. By permuting variables if necessary, observe that $S_1$ is of the form $x_{k+1} = x_{k+2} = \ldots = x_n = x_{n+1} > x_1, \ldots, x_m, x_{m+1}, \ldots, x_k$ and $S_2$ is of the form $x_{m+1} = \ldots = x_k = x_{l+k-m+1} = x_{l+k-m+2} = \ldots = x_n = x_{n+1} > x_1, \ldots, x_m, x_{k+1}, \ldots, x_{l+k-m}$. Thus, $S_1$ has $k$ free variables $\{x_1, \ldots, x_k\}$ and $S_2$ has $l$ free variables $\{x_1, \ldots, x_m, x_{k+1}, \ldots, x_{l+k-m}\}$, and they share the free variables $\{x_1, \ldots, x_m\}$. Hence $p \in S_1$ is of the form $p = (x_1, \ldots, x_k, x_0, \ldots, x_o)$ where there are $k$ $x_0$s and $q \in S_2$ is of the form $q = (y_1, \ldots, y_m, y_0, \ldots, y_0, y_1, \ldots, y_{l+k-m}, y_0, \ldots, y_0)$ where there are $k - m$ $y$s between $y_m$ and $y_{k+1}$ and $n + 1 - (l + k - m)$ $y$s after $y_{l+k-m}$, where $x_0 > x_1, \ldots, x_k$ and $y_0 > y_1, \ldots, y_m, y_{k+1}, \ldots, y_{l+k-m}$. Without loss of generality, assume $x_0 > y_0$, and let $d = y_0 - x_0$. Let $c = 0$, then $\max(\alpha, \beta q) \in S_3 \forall \alpha \leq 0$. Let $\beta = 0$, then $\max(\alpha, \beta q) \in S_3 \forall \beta = 0$. Let $\alpha > d$, $\max(\alpha, \beta q) \in S_1$ for $\alpha > d$, and $\max(\alpha, \beta q) \in S_2$ for $\alpha < d$. Thus, by definition of line segments, $[p, q] \subseteq S_1 \cup S_2 \cup S_3$ and $[p, q] \cap S_1 \neq \emptyset; \ i \in \{1, 2, 3\}$.

**Remark 4.7** As mentioned earlier, Lemma 4.6 shows that the line segment between any two boundary points goes through the face that is the union of those two points. Now consider the set $A = \{0, 1, \ldots, n\}$ and it’s powerset $P(A) = \{\emptyset, \{0\}, \ldots, \{1, 2, \ldots, n\}, A\}$, where $\{i\}$ implies that $x_i > 0, x_1, \ldots, x_n$ and $\{i, j, k\}$
implies that \( x_i = x_j = x_k > 0, x_1, \ldots, x_n, \) also \( x_0 = 0. \) This means that \( \emptyset \) is the whole space \( \mathbb{R}_{\text{max}}^n \) and \( A \) is the center, in our case the origin. Now since we have no reason to consider the entire space we introduce the set \( B = P(A)/\emptyset. \)

**Theorem 4.8** \( H \subseteq B \) is a hemispace iff \( H \) and \( B/H \) are closed under unions.

**Proof** Let \( H \) be a hemispace, this implies that \( H \) is convex and that all line segments in \( H \) remain in \( H. \) This means that if we take two points in \( H, \) by Lemma 4.6 and the fact that \( H \) is a hemispace, the union of the two planes that those points were on is in \( H. \) Since \( H \) is a hemispace the \( B/H \) is a hemispace, then \( B/H \) is also closed under unions.

Let \( H \) and \( B/H \) be closed under unions. Then for any two points in \( H \) the line segment between them stay's in \( H \) by Lemma 4.6. Similar for \( B/H. \) This implies that \( H \) and \( B/H \) are convex. Since \( H \) and \( B/H \) are convex, then they are both hemispaces.

**Remark 4.9** This theorem is not only true for a multi-section split but it can also be applied to a one section split. This gives all possibilities for all types of splits. In the next section we work on trying to count how many possible splits we have and how fast the number of splits grows as \( n \) grows.

### 5 Combinatorics

**Corollary 5.1** In \( \mathbb{R}_{\text{max}}^n \) there are exactly \( 2f(n) \) hemispaces at a finite point, where the function \( f \) is defined recursively by \( f(0) = 1 \) and

\[
f(n) = \left( \begin{array}{c} n+1 \\ 1 \end{array} \right) f(n-1) + \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) f(n-2) + \cdots + \left( \begin{array}{c} n+1 \\ n-1 \end{array} \right) f(1) + \left( \begin{array}{c} n+1 \\ n \end{array} \right) f(0) + 1. \quad (13)
\]

**Remark 5.2** The function \( f \) calculates the number of hemispaces that contain the origin. Thus, in order to calculate the total number of hemispaces we must multiply by 2 to count their complements as well.

**Proof** We will establish the corollary by induction on \( n. \) Clearly \( f(1) = 3 \) since the hemispaces in \( \mathbb{R}_{\text{max}}^n \) that contain the origin are the closed half-line to the right, closed half-line to the left, and the whole space. Now, suppose the formula holds for \( n-1 \) and we must show that it holds for \( n. \) Suppose \( H \) is a hemispace in \( \mathbb{R}_{\text{max}}^n \) containing the origin and \( G \) is its complement, and we will determine the total number of ways to construct \( H. \) Since the origin is in \( H, \) it follows that only one \( 1 - \text{face} \) can be in \( G \) since having any two \( 1 - \text{faces} \) in \( G \) would also require \( G \) to contain the origin to maintain convexity. Observe that there are

\[
\left( \begin{array}{c} n+1 \\ 1 \end{array} \right) 1 - \text{faces} \quad \text{in} \quad \mathbb{R}_{\text{max}}^n \quad \text{that can be in} \quad G.
\]

Once a \( 1 - \text{face} \) is chosen to be in \( G, \) then by the Theorem we know that every face containing the variable that is not in the \( 1 - \text{face} \) must be in \( H \) since having any of these faces in \( G \) would also require the origin to be in \( G \) to maintain convexity. Thus, finding the total number of hemispaces containing the origin now simplifies to an equivalent problem in \( \mathbb{R}_{\text{max}}^{n-1}, \) where \( G \) now contains the "origin" (the \( 1 - \text{face}. \) Hence, there are

\[
\left( \begin{array}{c} n+1 \\ 1 \end{array} \right) \quad \text{ways to reduce the problem to a} \quad n - 1 \quad \text{dimensional problem, but the number of hemispaces in} \quad \mathbb{R}_{\text{max}}^{n-1} \quad \text{is given by} \quad f(n-1), \quad \text{so this accounts for the}
\]

\[
\left( \begin{array}{c} n+1 \\ 1 \end{array} \right) f(n-1) \quad \text{term.}
\]

Alternatively, \( H \) could also contain every \( 1 - \text{face}. \) In this case, there can be no more than one \( 1 \) dimensional
in $G$ since, by the Theorem, putting any two $2$–faces in $G$ would also require either the origin or a $1$–face
to be in $G$ to maintain convexity. Thus, there are \( \binom{n+1}{2} \) $2$–faces that can be chosen to be in $G$, and
doing so simplifies the problem to the case in $\mathbb{R}^{n-2}_{\text{max}}$. This accounts for the \( \binom{n+1}{2} f(n-2) \) term. Again, we
could also choose for every $2$–face to be in $H$. Following this formula, it is clear that there are \( \binom{n+1}{d} \) ways to simplify the problem to the case of $\mathbb{R}^{n-d}_{\text{max}}$, which accounts for all the terms but the last "1", which
represents choosing to put every face in $H$, i.e. the trivial hemispace.

**Remark 5.3** Using this equation to find the number of hemispaces we get a sequence of numbers starting
at $f(0) = 1$. This sequence grows exponentially fast, it is also same sequences as the Ordered Bell Numbers
excluding the first term of that sequence.

## 6 Hemispaces that are not Semispaces

We know from [2] that the closure of a hyperplane is a hemispace. It is also clear that a hyperplane that
uses all variables available is a semispace or the boundary of a semispace. When each variable appears only
once we have the boundary of a hemispace.

**Example 6.1** A basic boundary in $\mathbb{R}^{2}_{\text{max}}$ is $x_1 \oplus x_2 = 0$. This is the boundary of the negative quadrant and
a semispace.

In this section we will look at what happens when we don’t use all of the variables, and what the boundaries
look like in this case. In $R^{2}_{\text{max}}$ we only have three hyperplanes that we can look at:

\[
x_1 = 0 \\
x_2 = 0 \\
x_1 = x_2
\]

These are very basic lines but they do form the boundary lines of hemispaces in $\mathbb{R}^{2}_{\text{max}}$.

**Remark 6.2** As with the rest of this paper we want to look at what the hemispaces of this form would look
like and how we could split the boundary, if at all, to have them remain hemispaces. It is very clear from
our example that if we have a hyperplane, like in the example, with only two variables. Then the boundary
if only one $n-1$–face. Since we have proved earlier that one point in the face being part of the boundary
means the whole face is in the boundary, then we can see that if there is only one face for the boundary,
then we only have the to trivial cases of the entire boundary being with one or the other sections.

**Definition 6.3** A hyperplane of the form:

\[
x_i \oplus \ldots \oplus x_m = x_k \oplus \ldots \oplus x_m; 0 \leq i < k \leq m < n
\]
is a hyperplane in $\mathbb{R}_{\max}^n$ with the boundary faces following the hyperplane:

$$x_i \oplus \ldots \oplus x_{k-1} = x_k \oplus \ldots \oplus x_m$$

and the section where $x_k, \ldots, x_m > x_i, \ldots, x_{k-1}$ is filled.

**Lemma 6.4** Let $x_0, x_1, \ldots, x_n$ be the variables in $\mathbb{R}_{\max}^n$, then any hemispaces (after permuting variables if necessary) in the form:

$$x_i \oplus x_{i+1} \oplus \ldots \oplus x_{i+m} = x_j \oplus x_{j+1} \oplus \ldots \oplus x_{j+t}$$

where $i + m < j$ and $m + t < n - 1$, then the hemispaces with this hyperplane as a border acts exactly like hemispaces in $\mathbb{R}_{\max}^{m+t+1}$ with sections that are the union of semispace complements.

**Remark 6.5** Since the space is going from $n \to m + t + 1$, then every dimensional space in the boundary of dimension $k$ is treated as if it is a $k - ((n + 1) - (m + t + 2)) - \text{face}$. Now that we know how a hyperplane with $2 \leq k \leq n + 1$ variables acts, it is important to know how many different hyperplanes there can be with $k$ variables in dimension $n$ ($f(k|n)$). The formula for that is:

$$f(k, n) = \binom{n+1}{2} \left( \frac{n-k+2}{1} \right) + \binom{n+1}{2} \left( \frac{n-k+3}{2} \right) + \ldots + \binom{n+1}{2} \left( \frac{n-1-k}{k-2} \right)$$

Where $f(2, n) = f(n+1, n) = \binom{n+1}{2}$.

### 7 Points at $-\infty$

After reordering the coefficients of the variables as needed, let $a_0 = a_1 = \ldots = a_l = 0 > -\infty = a_{l+1} = \ldots = a_n$. Then any hyperplane of the form:

$$a_0 x_0 \oplus \ldots \oplus a_m x_m \oplus a_{l+1} x_{l+1} \oplus \ldots \oplus a_k x_k = a_{m+1} x_{m+1} \oplus \ldots \oplus a_l x_l \oplus a_{k+1} x_{k+1} \oplus \ldots \oplus a_n x_n$$

With $m < l < k < n$, can be reduced to:

$$x_0 \oplus \ldots \oplus x_m = x_{m+1} \oplus \ldots \oplus x_l$$

Since $x_i \oplus a_s x_s = x_i$ with $i \in [0, \ldots, l]$ and $s \in [l+1, \ldots, n]$.

**Remark 7.1** In this case we have similar results to what we had in section 6. The main difference is, instead of all of the $k - \text{faces}$ dropping down in dimension, we now are only focusing on the $l$ dimensional cross section where $x_{l+1} = \ldots = x_n = -\infty$. This cross section follows the same rules with the splitting of the boundary as the rest of the paper did however, all of the higher dimensional faces that are not split have the ability to belong to either hemispace with no restrictions. Since these higher dimensional faces are not split and can belong to either hemispace, because that would still keep them convex by our earlier definition, we multiply our total number of hemispaces $2f(n)$ by two. So for this case we would have $4f(l)$ hemispaces, including complements.
Any hyperplane boundary of the form:

\[ a_1 x_1 \oplus ... \oplus a_l x_l \oplus a_{k+1} x_{k+1} \oplus ... \oplus a_n x_n = a_{l+1} x_{l+1} \oplus ... \oplus a_k x_k \]  

(19)

With \( l < k \leq n \), can reduce to:

\[ x_1 = x_2 = ... = x_l = ... = x_n = -\infty \]  

(20)

Since we have that \( x_1 \oplus ... \oplus x_l = \max(x_1, ..., x_l) = -\infty \).

**Remark 7.2** The hemispace in this situation would be \( x_1 \oplus ... \oplus x_l > -\infty \) in the \( l \) dimensional cross section with the point \(( -\infty, ..., -\infty )\) as the boundary point. Leaving only the trivial hemispace of the whole space without the point and the whole space with the point. Once again the space outside of the cross section does not split and can belong to either hemispace with effecting convexity.

The final situation is when all the coefficients are equal to \(-\infty\).

Then all of the variables are equal to \(-\infty\).

Recall from [1] that this situation will give us the semispace in \( \mathbb{R}^n_{\text{max}} \):

\[ S(-\infty) = \mathbb{R}^n_{\text{max}}/\{(-\infty, ..., -\infty)\}. \]  

(21)

The boundary of this semispace is simply the point \(( -\infty, ..., -\infty )\). Since we obviously can not split a point we are left with only two trivial hemispaces. Either this boundary point is with the semispace, and we have \( S(-\infty) \) and \( \bar{S}S(-\infty) \) as our two hemispaces.

Now consider the following hyperplane boundary:

\[ x_0 \oplus ... \oplus x_i \oplus a_{i+1} x_{i+1} \oplus ... \oplus a_m x_m = x_{m+1} \oplus ... \oplus x_l \oplus a_{l+1} x_{l+1} \oplus ... \oplus a_j x_j \]  

(22)

Where \( i < m < l < j \) and \( a_k = -\infty \ \forall k \)

This is a hyperplane boundary that is missing some variables and has some coefficients equal to \(-\infty\). First follow the rule for having coefficients equal to \(-\infty\) and take the \( n - ((m - i) + (j - l)) \) dimensional cross section where \( x_{i+1} = ... = x_m = x_{l+1} = ... = x_j = -\infty \). Then follow the rule for not having all the variables. So the cross section goes from acting like a \( \mathbb{R}^n_{\text{max}} \) space to a \( \mathbb{R}^n_{\text{max}} \) space.

### 8 References


