**TILLING A DEFICIENT RECTANGLE WITH T-TETROMINOES**

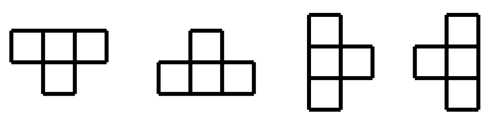
SHUXIN ZHAN

Abstract. In this paper, we will prove that no deficient rectangles can be tiled by T-tetrominoes.

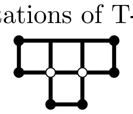
## 1. Introduction

The story of the mathematics behind tilling started with Solomon W. Golomb, who introduced and trademarked much of the relevant terminology [3][4]. An order $k$ polyomino is a two-dimensional shape consisting of $k$ unit squares joined along their edges. Modulo rotations and reflections, there exists one type of order 2 polyomino (domino), two types of order 3 polyominoes (trominoes), 5 types of order 3 polyominoes (tetrominoes), etc. A region, $R$, is **tileable** by a given set of tiles if it can be covered completely and without any overlap. An arrangement of tiles from a set that covers region $R$ completely and without overlap is called a **tilling** or **dissection**.

Out of the five possible configurations for tetrominoes, one is in the shape of a T, appropriately called a **T-tetromino** (shown in Figure 1). This shape has boundary length ten, six outer corners, and two inner corners.



(a) Four rotations of T-tetrominoes



(b) Inner (black dots) and outer corners (open dots) of a T-tetromino

Figure 1

In his 1965 paper [2], D.W. Walkup discovered that:

**Theorem 1.** *An $a \times b$ rectangle is tileable by T-tetrominoes if and only if $a$ and $b$ are both multiples of 4.*

Later, Korn and Pak in [5] explored the structure of tillings by T-tetrominoes by combining Walkup's inductive approach with defining a new height function and finding two bijections between this function and the tillings. They note the difficulty of finding a closed formula for the number of T-tetromino tillings of a rectangle. Subsequently, Merino in [1] found a closed-form formula for the number of tillings with T-tetrominoes for $4n \times 4m$ rectangles with $n$ =1,2,3, and 4, and also a computational method for values of $n \leq 8$.

In this paper, we will prove that it is impossible to tile any deficient rectangle (defined in Section 2) with T-tetrominoes. This result is surprisingly simple, especially considering the little progress on finding a closed formula for the number of ways to tile a regular rectangles by T-tetrominoes.

## 2. Deficient Rectangles

A **deficient rectangle** is a rectangle with sides of at least 2 and an unit square removed. T-tetrominos have area 4, so in order for an $m \times n$ deficient rectangle to be tileable, the area must satisfy $mn - 1 = 0 \pmod 4$, which occurs when $m \equiv n \equiv 1 \pmod 4$ or $m \equiv n \equiv 3 \pmod 4$. We represent any $m \times n$ rectangle on quadrant I of the cartesian plane with the lines $y = 0$, $y = m$, $x = 0$, and $x = n$. Associate each square in the rectangle with the coordinate $(x, y)$ where $(x, y)$ lies at the bottom left corner of the square.

Consider a region $R$. A **segment** is a line segment of length 1 forming the edge of a unit square. A segment is a **cut** if, in every dissection of $R$, it lies on one of the ten boundary segments of some T-tetromino. Note that every segment on the boundary of a deficient rectangle and around the missing square is a cut segment.

A point is called **cornerless** if in every tilling of $R$, it does not lie on any of the six outside corners of a T-tetromino. A point is called **inner cornerless** if it does not lie on any of the two inside corners of any T-tetromino in a tilling of $R$.

Note that in any tilling of a rectangle with T-tetrominoes, if a point is an inner corner of some T-tetromino, then in order for the square adjacent to the point to be tiled, the point must also lie on an outer corner. Thus, a cornerless point is inner cornerless, which means it cannot lie on any corner of a tetromino, regardless of whether it is inner or outer. In a deficient rectangle, we can cut the rectangle such that every point except for the points surrounding the missing square is inside a nondeficient rectangle. Therefore, a cornerless point in a deficient rectangle that does not surround the missing square is inner cornerless.

A **translate** of a point, segment, or T-tetromino is another point, segment, or T-tetromino in the quadrant obtained from a displacement of $2k$ in $y$ and $-2k$ in $x$, where $k$ is any integer.

Consider an $a \times b$ rectangle on quadrant I. A point is called **type-$\boldsymbol{A_1}$** if its coordinates are congruent to $(0, 0) \pmod 4$ or $(2, 2) \pmod 4$ and **type-$\boldsymbol{B_1}$** if its coordinates are congruent to $(0, 2) \pmod 4$ or $(2, 0) \pmod 4$. Furthermore, a point is called **type-$\boldsymbol{A_2}$** if its coordinates are congruent to $(a, b) \pmod 4$ or $(a-2, b-2) \pmod 4$ and **type-$\boldsymbol{B_2}$** if its coordinates are congruent to $(a, b-2) \pmod 4$ or $(a - 2, b) \pmod 4$.

Furthermore, any translate of a type-$A_1$, type-$A_2$, type-$B_1$, and type-$B_2$ is another point of the same type. For a deficient $m \times n$ rectangle where $m \equiv n \equiv 1 \pmod 4$ or $m \equiv n \equiv 3 \pmod 4$, all type-$A_2$ points are $(1, 1) \pmod 4$ or $(3, 3) \pmod 4$ and all type-$B_2$ points are $(1, 3) \pmod 4$ or $(3, 1) \pmod 4$.

**Lemma 1.** *Let $m \times n$ be a deficient rectangle with the square missing at $(x_0, y_0)$. Then,*

*(1) every type-$B_1$ point on or below the line $x + y = 4\lfloor \frac{x_0+y_0}{4} \rfloor$ is cornerless and each of the 2,3, or 4 segments incident to a type-$A_1$ point and below $x + y = 4\lfloor \frac{x_0+y_0}{4} \rfloor$ is a cut.*

*(2) every type-$B_2$ point on or above the line $x + y = 4\lceil \frac{x_0+y_0-1}{4} \rceil + 2$ is cornerless and each of the 2,3, or 4 segments incident to a type-$A_2$ point and above $x + y = 4\lceil \frac{x_0+y_0-1}{4} \rceil + 2$ is a cut.*

In order to prove this result, we reference a lemma from Walkup's paper [2]:

**Lemma 2.** *Define a point to be **type-A** if it is congruent to $(0, 0)$ or $(2, 2) \pmod 4$ and **type-B** if it is congruent to $(0, 2)$ or $(2, 0) \pmod 4$. For a $m \times n$ rectangle on quadrant I, every type-B point is cornerless and each of the 2, 3, or 4 segments incident on a type-A point is a cut (Figure 2).*

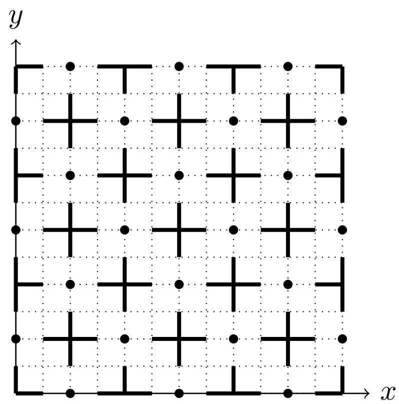



FIGURE 2. type-A (intersection of black lines) and type-B (black dots)

*Proof of Lemma 1.* Walkup's proof of Lemma 2 works inductively on the diagonals of the rectangle: for $\lambda \in \mathbb{N}$, let $P(\lambda)$ be the proposition that the lemma holds for all type-A and type-B points on or below the line $x + y = 4\lambda$. We can use a similar argument for deficient rectangles, but only up to values of $\lambda$ where $x + y = 4\lambda$ does not intersect or go beyond the missing square, because the missing square may interfere with the inductive step. Note that type-$A_1$ and type-$B_1$ are analogous to Walkup's definition of type-A and type-B points. The first line which intersects or go beyond the missing square is $x + y = 4\lfloor \frac{x_0+y_0}{4} \rfloor$. Therefore, the results from Walkup's Lemma holds for type-$A_1$ and type-$B_1$ points on and below this line.

Similarly, we can rotate the board such that the top right corner is at the origin. Through this, we can apply Walkup's Lemma to type-$A_2$ and type-$B_2$ points on and above the line $x + y = 4\lceil \frac{x_0+y_0-1}{4} \rceil + 2$ (Figure 3).

$\square$

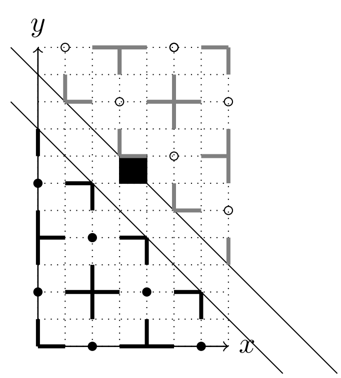


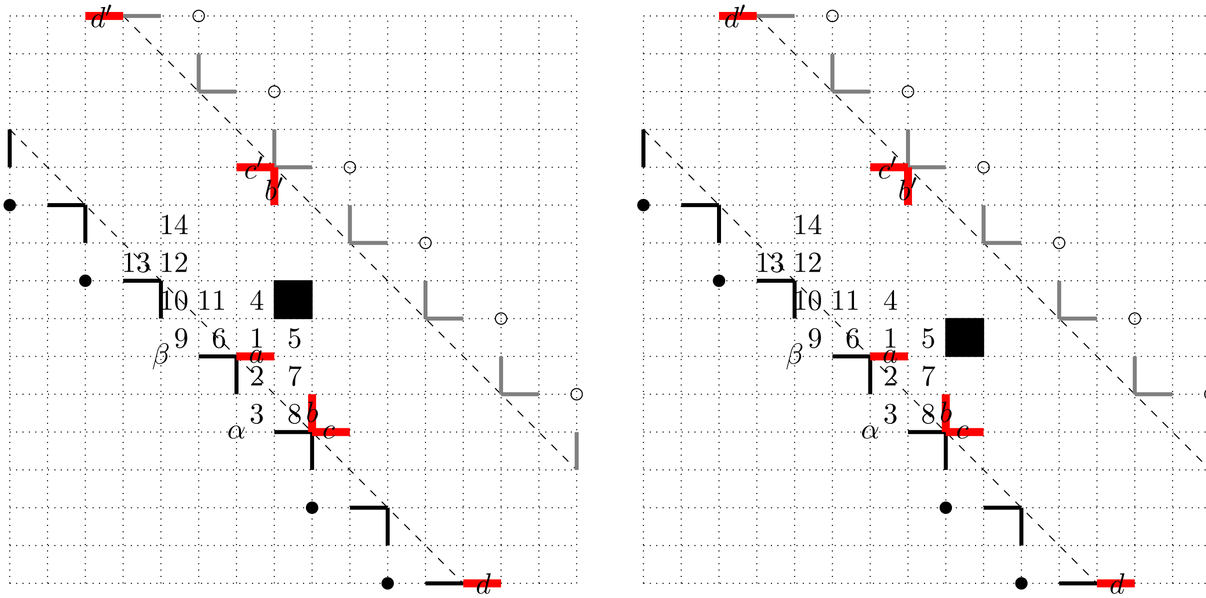

FIGURE 3. $7 \times 11$ deficient rectangle with type-$A_1$ (black dots), $A_2$ (open dots), $B_1$ (intersection of black lines), and $B_2$ (intersection of grey lines) points shown.

**Theorem 2.** *Any deficient $m \times n$ rectangle is not tileable by T-tetrominoes.*

*Proof.* Let $(x_0, y_0)$ be the missing square. Note that in the following figures, certain squares will be numbered. We will refer to a T-tetromino by a sequence of four numbers, such as "1-2-3-4", such that each number will represent one of the squares that make up the T-tetromino.

Proceed by cases:

**Case 1:** $(x_0, y_0)$ is $(1,1)$, $(3,3)$, $(0,2)$, or $(2,0) \pmod 4$




(A) $(x_0, y_0) = (1,1)$ or $(3,3) \pmod 4$  (B) $(x_0, y_0) = (0,2)$ or $(2,0) \pmod 4$

FIGURE 4. Borders $b$, $c$, $b'$, $c'$ and all their translates (highlighted in red) are cuts

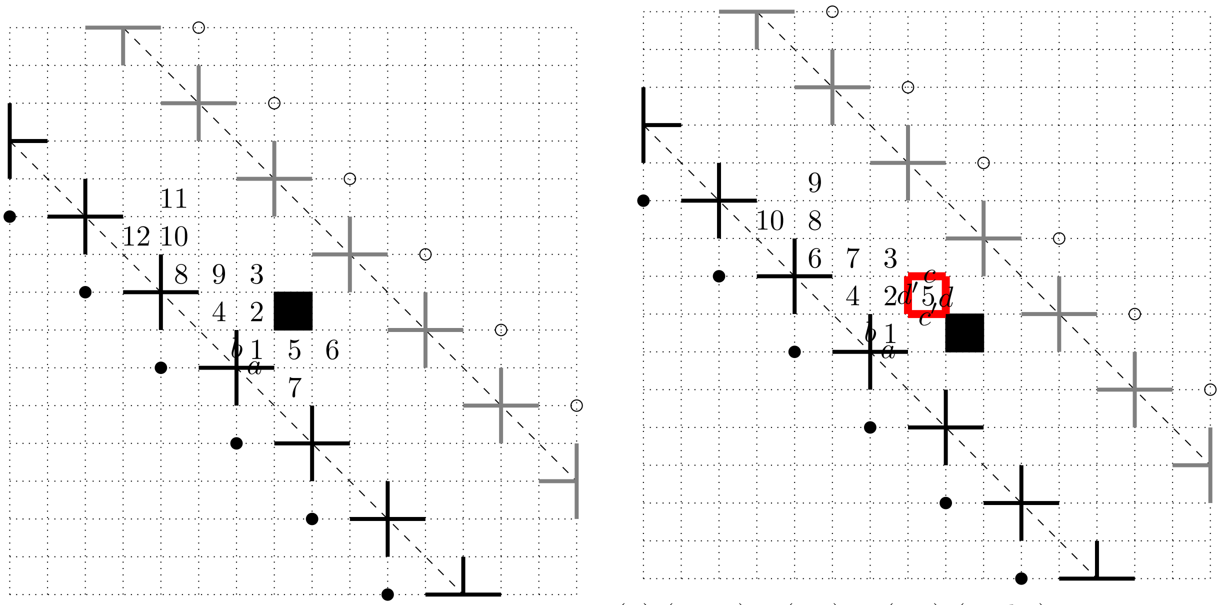
First, we want to prove that segments $b$, $c$, $b'$, $c'$and their translates are cuts (Figure 4A and 4B). By symmetry, we only need to show that $c$ and its translates are cuts. Because the border segment, $d$, must be a cut, it is sufficient to prove that if $c$ is a cut, then its adjacent translate, $a$, is also a cut.

Assume that $c$ is a cut but $a$ is not a cut. Therefore, there exists a tilling which has a tetromino that contains both square 1 and 2. This tetromino cannot contain 3, because $\alpha$ is cornerless. Therefore, it must contain two out of three squares, 4, 5, and 6. If it contained 5, then there are no ways to tile 7 and 8. Therefore, 1-2-4-6 must be contained in the dissection. Because $\beta$ is cornerless, it is also inner cornerless, and so squares 9 and 10 must be contained in two different tetrominoes. To tile square 10, the dissection must contain 10-12-13-14, otherwise tilling square 11 would be impossible. By repeating the same argument, all upward translates of 1-2-4-6 must be included in the dissection. However, the translates will eventually reach the boundaries of the rectangle, making tilling impossible. Thus if $c$ is a cut, then so is $a$. Since the border segment $d$ is a cut, all of its translates are all cuts. By symmetry, $c$, $b'$, $c'$and their translates are also all cuts.

We now consider the case where the missing square $(x_0, y_0) = (1,1)$ or $(3,3) \pmod 4$ (Figure 5A). The only way to tile square 1 is by 1-2-3-4 or 1-5-6-7. If the tilling contained 1-2-3-4, then it must also contain 8-10-11-12, because that is the only way for both square 8 and 9 to be tileable. By induction, all upward translates of 1-2-3-4 must be in the tilling, but this is impossible because of the $y$-axis. Thus, 1-2-3-4 is not in the tilling, and by symmetry, neither is 1-5-6-7. This proves that tilling is impossible for $(x_0, y_0) = (1,1)$ or $(3,3) \pmod 4$.

Lastly, we consider the case where $(x_0, y_0) = (0,2)$ or $(2,0) \pmod 4$ (Figure 5B). We claim that segments $c$ and $d$ must be cuts, and by symmetry, $c'$ and $d'$ would also be cuts.

Suppose $c$ is not a cut, then the only way to cover 1 such that $c$ is not on a boundary is by 1-2-3-4. Similarly, suppose $d$ is not a cut, then the only way to cover square 1 such that $d$ is not on a




(A) $(x_0, y_0) = (1, 1)$ or $(3, 3) \pmod 4$

(B) $(x_0, y_0) = (0, 2)$ or $(2, 0) \pmod 4$. (Segments $c$, $c'$, $d$, $d'$ are highlighted in red.)

FIGURE 5. tilling is not possible

boundary is also by 1-2-3-4. Thus if $c$ or $d$ are not cuts, then there must exist a dissection which contains the T-tetromino 1-2-3-4. In order for squares 6 and 7 of this dissection to be tileable, the tetromino 6-8-9-10 must be in the tilling. By induction, all upward translates of 1-2-3-4 must be in the tilling. This is impossible, so both $c$ and $d$ must be cuts, and by symmetry, $c'$ and $d'$ are also cuts. This is impossible, because the four segments surrounding square 5 are all cut segments, so there is no way of tilling 5. This proves that tilling is impossible for $(x_0, y_0) = (0, 2)$ or $(2, 0) \pmod 4$.

**Case 2:** $(x_0, y_0)$ is $(0, 0)$ or $(2, 2) \pmod 4$

The cuts from Lemma 1 are as shown in Figure 6.

By contradiction, first assume $a$ is not a cut. Consider the ways to tile square 1. Tetrominoes 1-2-3-4 and 1-2-5-6 cannot be in the tilling because points $\alpha$ and $\beta$ are cornerless. This leaves 1-3-7-8, 1-3-5-8, 1-3-5-7, and 1-5-7-8. Since $a$ is not a cut, there must exist a tilling of the rectangle in which $a$ is not on a boundary segment, so there must exist a tilling that includes 1-5-7-8. This tilling must also contain 2-3-9-15, since this is the only way to tile square 3 without $a$ being on the boundary of a T-tetromino. The only ways to tile square 10 without intersecting the nearby cuts are by 10-11-12-13 or 10-13-14-16, but both are not possible, because $\gamma$ and $\delta$ are cornerless. This is a contradiction, so $a$ must be a cut, and by symmetry, $b$ is also a cut.

Because $a$ and $b$ are cuts, the only ways of tilling square 2 is by 2-9-10-11 or 2-10-15-16, but $\gamma$ and $\delta$ are cornerless, so there are no possible ways to tile this deficient rectangle.

**Case 3:** $(x_0, y_0)$ is $(1, 3)$ or $(3, 1) \pmod 4$

Consider Figure 7, which is a generic representation of a rectangle missing square $(x_0, y_0) = (1, 3)$ or $(3, 1) \pmod 4$. To tile square 1, a dissection of the rectangle must contain T-tetrominoes 1-2-3-4, 1-2-3-5, 1-6-7-8, 1-6-7-9, or 1-7-8-9. Because $\alpha$ is cornerless, 1-7-8-9 cannot be in the dissection.

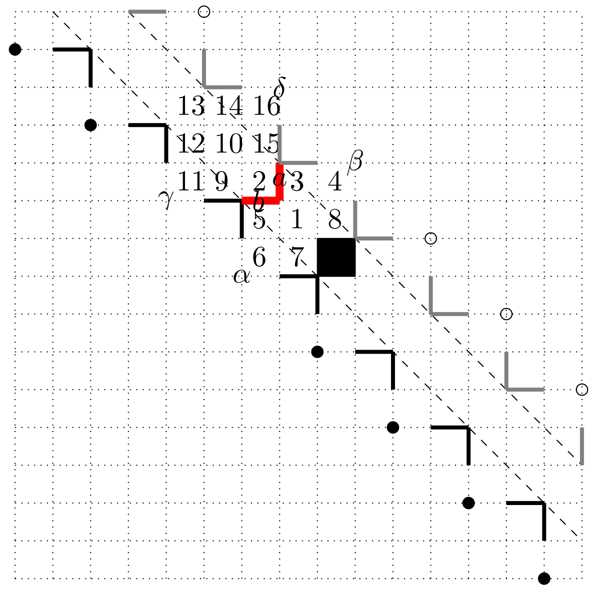


FIGURE 6. tilling is not possible for $(x_0, y_0) = (0,0)$ or $(2,2)$ (mod 4): Segments $a$ and $b$ are highlighted in red.

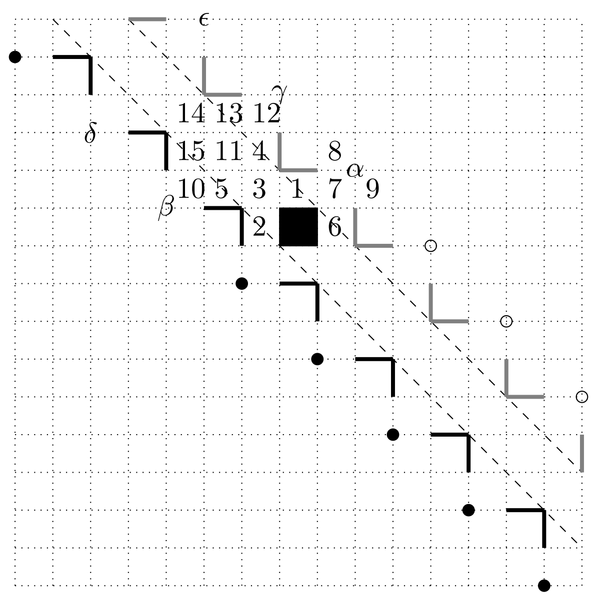
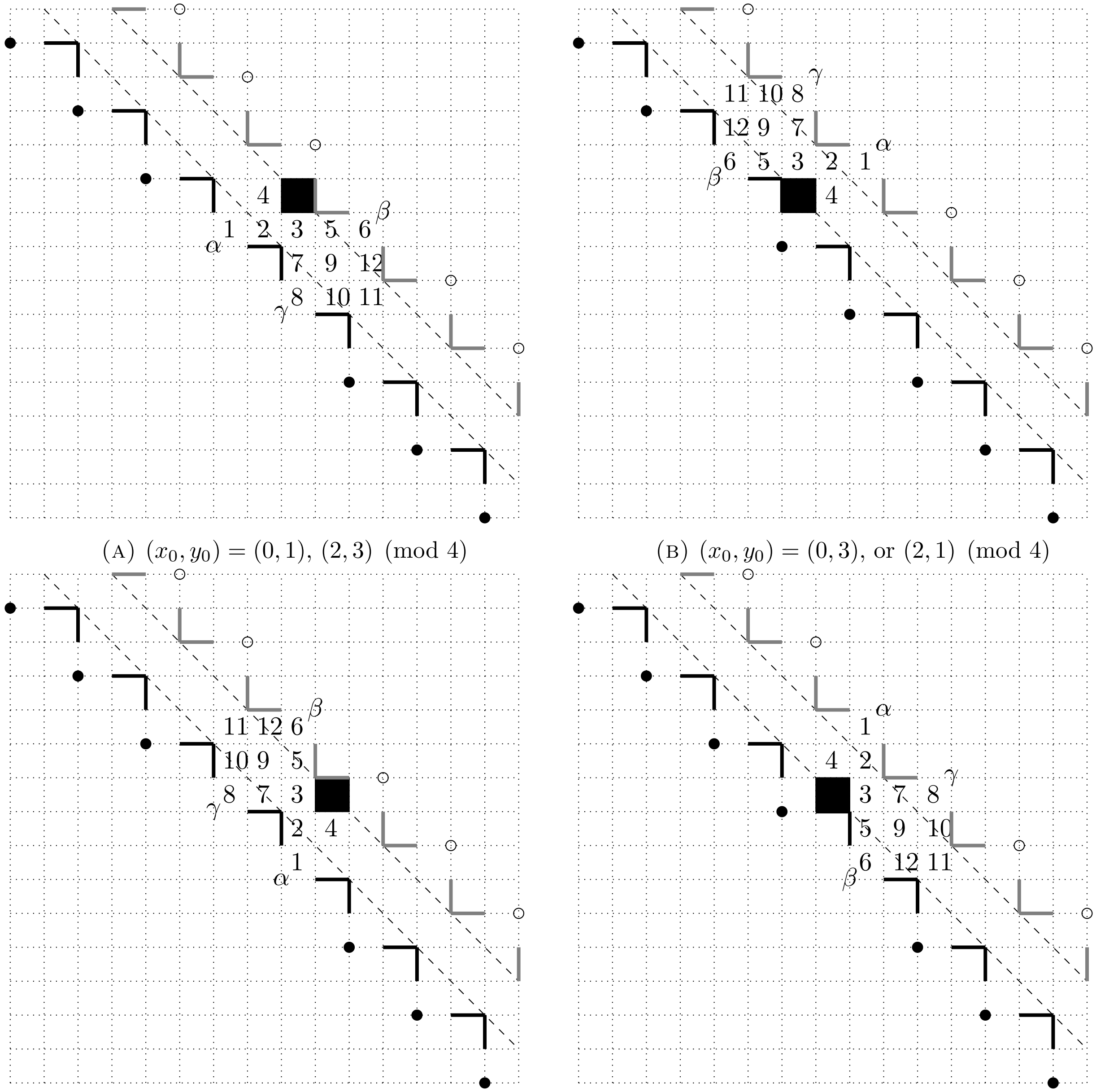


FIGURE 7. tilling is not possible for $(x_0, y_0) = (1,3)$ or $(3,1)$ (mod 4)

Suppose 1-6-7-8 or 1-6-7-9 are in the dissection. To tile square 3, either 2-3-4-5, 3-5-10-11, or 3-4-11-12 need to be in the dissection. $\beta$ and $\gamma$ are cornerless, so 3-5-10-11 and 3-4-11-12 are not possible. If 2-3-4-5 is in the dissection, then there is no way of tilling square 11, because 10-11-14-15 and 11-12-13-14 are needed to cover square 11, but $\beta$ and $\gamma$ are cornerless. Thus neither 1-6-7-8 nor 1-6-7-9 can be in the dissection of the rectangle.

Suppose 1-2-3-5 is in the dissection. Because $\beta$ and $\gamma$ are cornerless, the only way to tile square 11 is by 4-11-13-15. Then there are no ways of tilling square 14, because $\delta$ and $\epsilon$ are cornerless. Thus 1-2-3-5 cannot be in any dissection of the rectangle, and by symmetry, neither can 1-2-3-4. Thus tilling is not possible for $(x_0, y_0) = (1,3)$ or $(3,1)$ (mod 4).

**Case 4:** $(x_0, y_0)$ is $(0,1)$ or $(1,0)$ (mod 4)




(A) $(x_0, y_0) = (0,1)$, $(2,3)$ (mod 4)

(B) $(x_0, y_0) = (0,3)$, or $(2,1)$ (mod 4)

(C) $(x_0, y_0) = (1,0)$, $(3,2)$ (mod 4)

(D) $(x_0, y_0) = (3,0)$, or $(1,2)$ (mod 4)

FIGURE 8. tilling is not possible

If $(x_0, y_0) = (0,1)$ or $(1,0)$ (mod 4), then there are four possibilities for the relative location between the cuts and cornerless points from lemma 1 and the missing square, shown in Figure 6. It is clear from Figure 6 that these four cases are rotationally symmetric. The labeling have been rotated so that the following proof holds for all four figures:

In order to tile square 3, a dissection of the rectangle must contain T-tetrominoes 1-2-3-4, 3-5-6-9, 3-7-8-9, or 2-3-5-7. Because $\alpha$, $\beta$, and $\gamma$ are cornerless, T-tetrominoes 1-2-3-4, 3-5-6-9, and 3-7-8-9 cannot be in the dissection, which leaves 2-3-5-7. If a tilling contained 2-3-5-7, then the only ways or tilling square 9 is by 6-9-11-12 or 8-9-10-11, but both are not possible, because $\beta$ and $\gamma$ are cornerless. Thus tilling is impossible.

$\square$

## 3. Conclusion

In this paper, we proved that no deficient rectangle can be tiled using T-tetrominoes. The proof used the inductive argument on diagonals introduced by Walkup [2] to reduce the problem into four cases. We finished the proof by showing each case produce contradictions.

In Walkup's work [2] and this paper, using induction by diagonal produced a structure of points and segments in the rectangle with special properties. It would be interesting to study whether we can apply this technique to other families of tiles, and whether the applicability of this technique reflects anything about the family itself.

## 4. Acknowledgments

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