Tiling a Deficient Rectangle with T-Tetrominoes

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A **tetromino** is a two-dimensional shape formed by connecting 4 unit squares along their edges.

**T-tetromino** is a tetromino in the shape of a "T".

- A region, \( R \), is **tileable** by a given set of tiles if it can be covered completely and without any overlap.
- A **deficient rectangle** is a positive integer dimensioned rectangle with sides of at least 2, and a 1x1 unit square removed.
- In his 1965 paper, D.W. Walkup discovered that a \( a \times b \) rectangle is tillable by T-tetrominoes if and only if \( a \) and \( b \) are both multiples of 4.
T-tetrominos have area 4, so in order for an $m \times n$ deficient board to be tileable, $m \equiv n \equiv 1 \pmod{4}$ or $m \equiv n \equiv 3 \pmod{4}$. 

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For any board, we represent it on quadrant I of the cartesian plane with the lines $y = 0$, $y = m$, $x = 0$, and $x = n$. Associate each square in the board with the coordinate $(x, y)$ where $(x, y)$ is the point at the bottom left corner of the square.

- A **segment** is a line segment of length 1 forming the edge of a unit square of the quadrant.
- A segment is a **cut** if in every dissection of the quadrant it is one of the ten boundary segments of some T-tetromino.
- A point is **cornerless** if it does not lie at any of the six outside corners of any T-tetromino in any tiling of the board and **inner cornerless** if it does not lie at any of the two inside corners of any T-tetromino in any tiling of the board.
  - In a tiling of a rectangle with T-tetrominoes, inner corner $\implies$ outer corner, and so cornerless point $\implies$ inner cornerless.
  - In a deficient board, we can cut the board such that:
    - a cornerless point not surrounding the missing square $\implies$ inner cornerless.
- A **translate** of a point, segment, or T-tetromino is another point, segment, or T-tetromino in the quadrant obtained from a displacement of $2k$ in $y$ and $-2k$ in $x$, where $k$ is any integer.
Consider an $m \times n$ board on quadrant I. A point is called type-A if it is congruent to $(0, 0)$ or $(2, 2)$ (mod 4). It is called type-B if it is congruent to $(0, 2)$ or $(2, 0)$ (mod 4).

**Lemma**

*Every type-B point is cornerless and each of the 2, 3, or 4 segments incident on a type-A point is a cut.*

**Sketch of Proof.**

For $\lambda \in \mathbb{N}$, let $P(\lambda)$ be the proposition that the lemma holds for all type-A and type-B points on or below the line $x + y = 4\lambda$. $P(0)$ is obviously true, because the segments incident on the origin must be cuts. Assume $P(\lambda)$ holds, the rest of Walkup’s paper shows that this implies $P(\lambda + 1)$ also holds.
Walkup’s Lemma

Figure: $12 \times 12$ board with type-A (intersection of black lines) and type-B (black dots) points shown.
Consider an \( m \times n \) board on quadrant I. A point is called:

- **type-A\(_1\)** if its coordinates are congruent to \((0, 0)\) or \((2, 2)\) (mod 4)
- **type-B\(_1\)** if its coordinates are congruent to \((0, 2)\) or \((2, 0)\) (mod 4).
- **type-A\(_2\)** if its coordinates are congruent to \((m, n)\) or \((m - 2, n - 2)\) (mod 4)
- **type-B\(_2\)** if its coordinates are congruent to \((m, n - 2)\) or \((m - 2, n)\) (mod 4).

Any translate of a type-A\(_1\), type-A\(_2\), type-B\(_1\), and type-B\(_2\) is another point of the same type.

For a deficient \( m \times n \) board where \( m \equiv n \equiv 1 \) (mod 4) or \( m \equiv n \equiv 3 \) (mod 4), all type-A\(_2\) points are \((1, 1)\) (mod 4) or \((3, 3)\) (mod 4) and all type-B\(_2\) points are \((1, 3)\) (mod 4) or \((3, 1)\) (mod 4).

**Lemma**

Let \( m \times n \) be a deficient board with the square missing at \((x_0, y_0)\). Then,

1. every type-B\(_1\) point below the line \( x + y = 4 \left\lfloor \frac{x_0 + y_0}{4} \right\rfloor \) is cornerless and each of the 2,3, or 4 segments incident on a type-A\(_1\) point and below \( x + y = 4 \left\lfloor \frac{x_0 + y_0}{4} \right\rfloor \) is a cut.

2. every type-B\(_2\) point above the line \( x + y = 4 \left\lceil \frac{x_0 + y_0 - 1}{4} \right\rceil + 2 \), is cornerless and each of the 2,3, or 4 segments incident on a type-A\(_2\) point and above \( x + y = 4 \left\lceil \frac{x_0 + y_0 - 1}{4} \right\rceil + 2 \) is a cut.
Figure: $7 \times 11$ deficient board with type-$A_1$ (black dots), $A_2$ (open dots), $B_1$ (intersection of black lines), and $B_2$ (intersection of grey lines) points shown.
Figure: Tiling is not possible for \((x_0, y_0) = (0, 0)\) or \((2, 2) \text{ (mod 4)}\)
Claim

Segments a and b are cuts.

Proof.

By contradiction, first assume a is not a cut. Consider the ways to tile square 1. Tetrominoes 1-2-3-4 and 1-2-5-6 cannot be in the tiling because points $\alpha$ and $\beta$ are cornerless. This leaves 1-3-7-8, 1-3-5-8, 1-3-5-7, and 1-5-7-8. Since $a$ is not a cut, there must exist a tiling of the board in which $a$ is not on a boundary segment, so there must exist a tiling that includes 1-5-7-8. This tiling must also contain 2-3-9-15, since this is the only way to tile square 3 without $a$ being on the boundary of a tile. The only ways to tile square 10 without intersecting the nearby cuts are by 10-11-12-13 or 10-13-14-16, but both are not possible, because $\gamma$ and $\delta$ are cornerless. This is a contradiction, so $a$ must be a cut, and by symmetry, $b$ is also a cut.

The only ways of tiling square 2 without intersecting $a$ or $b$ is by 2-9-10-11 or 2-10-15-16, but $\gamma$ and $\delta$ are cornerless, so there are no possible ways to tile this deficient board.
Claim: Segments $a$ and $b$ are cuts.
Assume $a$ is not a cut and consider the ways to tile square 1.

**Figure:** Assume $a$ is not a cut. There must exist some tiling of the board in which $a$ is not a boundary. Consider the ways to tile square 1.
Figure: Tetrominoes 1-2-3-4 and 1-2-5-6 cannot be in the tilling because points $\alpha$ and $\beta$ are cornerless.
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Figure: The only ways to tile square 10 without intersecting the nearby cuts are by 10-11-12-13 or 10-13-14-16, but both are not possible, because $\gamma$ and $\delta$ are cornerless.
Segments $a$ and $b$ are cuts

Figure: The only ways of tiling square 2 without intersecting $a$ or $b$ is by 2-9-10-11 or 2-10-15-16, but $\gamma$ and $\delta$ are cornerless, so there are no possible ways to tile this deficient board.