Some research directions
on one-dimensional nonlinear hyperbolic equations

(Alberto Bressan, July 2016)

1 Solutions with large BV data

For hyperbolic systems of conservation laws, in the past decades a satisfactory theory has been established for solutions with small total variation [1, 3, 14, 19]. This includes global existence and uniqueness of solutions to the Cauchy problem, Lipschitz continuous dependence on initial data w.r.t. the $L^1$ distance, and convergence of vanishing viscosity approximations.

A major issue of current research is whether this theory remains valid solutions with large BV data.

Jenssen’s counterexample [15] shows that, for some special systems of conservation laws, the total variation of solutions can blow up in finite time. However, it is not yet clear if this blow up can actually occur also for physical systems, such as the Euler equations of inviscid gas dynamics, endowed with a strictly convex entropy.

In particular, the $2 \times 2$ system describing isentropic gas dynamics was considered in 1860 by B. Riemann [17], who constructed a famous class of solutions with piecewise constant initial data. Even for this basic system, the global existence of solutions with large total variation has remained a challenging open problem.

In Lagrangean variables the Cauchy problem for the $p$-system of isentropic gas dynamics takes the form

\[
\begin{cases} 
  v_t - u_x = 0, \\
  u_t + p(v)_x = 0,
\end{cases} \quad \begin{cases} 
  v(0, x) = v_0(x), \\
  u(0, x) = u_0(x).
\end{cases}
\]  

(1.1)

Here $u$ is the velocity, $\rho$ is the density, $v = \rho^{-1}$ is specific volume, while $p = p(v)$ is the pressure.

**Question 1:** Consider the pressure law $p(v) = v^{-\gamma}$, for some $1 < \gamma < +\infty$. Let $(v, u)$ be an entropy-admissible solution of the Cauchy problem (1.1). Assume that the initial data has bounded variation:

\[ \text{Tot.Var.}\{v_0\} + \text{Tot.Var.}\{u_0\} = M_0 < +\infty, \]

and that the density remains bounded below at all times, so that

\[ \frac{1}{\rho(t, x)} = \frac{1}{\rho(t, x)} = v(t, x) \leq C \quad \text{for all } t > 0, \ x \in \mathbb{R}. \]

Does this imply that the solution has a uniformly bounded total variation at all times $t > 0$? In other words, does there exists a constant $M$ depending only on $M_0, C$, such that

\[ \text{Tot.Var.}\{v(t, \cdot)\} + \text{Tot.Var.}\{u(t, \cdot)\} = M \]

for all $t > 0$?
The recent examples in [4, 5] show that, for a large class of pressure laws \( p(v) \), one can construct \textbf{approximate solutions} by front-tracking whose total variation grows without bounds. As a consequence, a priori BV bounds for \textbf{exact solutions} cannot be established simply by estimating wave strengths across interactions. If such bounds can ever be proved, the decay of rarefaction waves due to genuine nonlinearity must also be taken into account.

A related issue is the possible emergence of vacuum in finite time. For smooth solutions, the analysis in [16] proves that the density can approach zero as \( t \to +\infty \), but vacuum is never reached in finite time. A major open problem is whether the same result holds for general BV solutions, possibly containing shocks.

2 Generic singularities for nonlinear wave equations

For a large class of nonlinear wave equations, solutions with smooth initial data can lose regularity within finite time. Typically, this can be proved by showing that the norm

\[ \|u(t, \cdot)\|_{C^1(\mathbb{R})} \]

blows up in finite time.

Establishing that singularities do occur is only a first step in a more detailed analysis. A general research program in this direction can be outlined as follows:

- Prove (or disprove) that, for “generic” smooth initial data, singularities are localized along finitely many points, or curves, in the \( t-x \) plane.
- Give a local asymptotic description of these generic singularities.

Here we say that a property is “generic” if it holds true on a countable intersection of open dense sets in \( C^k(\mathbb{R}) \).

In principle, the above research program may apply to a wide variety of nonlinear wave equations. Here we consider three main examples.

2.1 Hyperbolic systems of conservation laws:

\[ u_t + f(u)_x = 0. \quad (2.2) \]

For scalar conservation laws with convex flux, a famous result by Schaeffer [18] shows that, for generic initial data in \( C^3(\mathbb{R}) \), the solution is piecewise regular, with at most a locally finite set of shocks.

As proved in the recent paper [13], this generic regularity result does not extend to \( 3 \times 3 \) systems. In this direction, the major remaining open problem is what happens for \( 2 \times 2 \) systems.

\textbf{Question 2.} Let (2.2) be a strictly hyperbolic, genuinely nonlinear \( 2 \times 2 \) system of conservation laws. Is it true that, for a generic smooth initial data \( u(0, \cdot) = u_0 \in C^3(\mathbb{R}; \mathbb{R}^2) \) (with small total variation) the solution is piecewise smooth with a locally finite set of shock curves?
2.2 The Burgers-Hilbert equation.

The integro-differential equation

\[ u_t + \left( \frac{u^2}{2} \right)_x = H[u] \quad (2.3) \]

was derived in [2] as a model for nonlinear waves with constant frequency. Here

\[ H[f](x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, dy \]

denotes the Hilbert transform of a function \( f \in L^2(\mathbb{R}) \). Global existence of solutions with arbitrary initial data in \( L^2(\mathbb{R}) \) was proved in [10], while uniqueness remains largely an open question. Piecewise smooth solutions were constructed in [12], providing a detailed description of the behavior near a shock.

The generic regularity of solutions to the Burgers-Hilbert equation is still an open problem.

2.3 Variational wave equations.

For conservative solutions of the second order wave equations of the form

\[ u_{tt} - c(u)(c(u)u_x)_x = 0, \quad (2.4) \]

a fairly complete existence-uniqueness theory is now available [11, 4]. This applies to general initial data

\[ u(0, \cdot) = u_0 \in H^1(\mathbb{R}), \quad u_t(0, \cdot) = u_1 \in L^2(\mathbb{R}). \]

The recent analysis in [7] shows that a generic conservative solution is smooth outside a locally finite set of curves where the gradient blows up. A detailed description of the structure of solutions in a neighborhood of every generic singularity can be found in [8].

On the other hand, for dissipative solutions both the uniqueness and the generic regularity remain open questions. See [9] for a constructive method that may be useful in this direction.

References


