An Introduction to the Mathematical Theory of Nonlinear Control Systems

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1. Definitions and examples of nonlinear control systems
2. Relations with differential inclusions
3. Properties of the set of trajectories
4. Optimal control problems
5. Existence of optimal controls
6. Necessary conditions for optimality: the Pontryagin Maximum Principle
7. Viscosity solutions of Hamilton-Jacobi equations
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Control Systems

\[ \dot{x} = f(x, u) \quad x \in \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m \]  
\( x(0) = x_0 \) \hspace{1cm} (1)

A trajectory of the system is an absolutely continuous map \( t \mapsto x(t) \) such that there exist a measurable control function \( t \mapsto u(t) \in U \), such that \( \dot{x}(t) = f(x(t), u(t)) \) for a.e. \( t \in [0, T] \).

Basic assumptions

1 - Sublinear growth

\[ |f(x, u)| \leq C \left(1 + |x|\right) \quad u \in U, \quad x \in \mathbb{R}^n \]

guarantees that solutions remain uniformly bounded, for \( t \in [0, T] \).

2 - Lipschitz continuity

\[ |f(x, u) - f(y, u)| \leq L |x - y| \quad u \in U, \quad x, y \in \mathbb{R}^n \]

if \( x(\cdot), \ y(\cdot) \) are trajectories corresponding to the same control function \( u(\cdot) \), this implies

\[ |x(t) - y(t)| \leq e^{Lt} |x(0) - y(0)| \]

hence the Cauchy problem (1)-(2) has a unique solution.

Equivalent Differential Inclusion

\[ \dot{x} \in G(x) \doteq \{ f(x, u) ; \ u \in U \} \]  
(3)

A trajectory of (3) is an absolutely continuous map \( t \mapsto x(t) \) such that \( \dot{x}(t) \in G(x(t)) \) for almost every \( t \in [0, T] \).
Optimization problems

Select a control function $u(\cdot)$ that performs an assigned task *optimally*, w.r.t. a given cost criterion.

Open-loop control

$u = u(t)$ is a (possibly discontinuous) function of time. Yields an O.D.E. of the form

$$\dot{x}(t) = g(x, t) \triangleq f(x, u(t))$$

where $g$ is Lipschitz continuous in $x$, measurable in $t$.

For solutions of the O.D.E., standard existence and uniqueness results hold, for Caratheodory solutions

Closed-loop (feedback) control

$u = u(x)$ is a (possibly discontinuous) function of space. Yields an O.D.E. of the form

$$\dot{x}(t) = g(x) \triangleq f(x, u(x))$$

where $g$ is measurable, possibly discontinuous.

For solutions of the O.D.E., no general existence and uniqueness result is available
1 - Navigation Problem

\[ x(t) = \text{position of a boat on a river} \]

\[ \mathbf{v}(x) = \text{velocity of the water} \]

Control system

\[ \dot{x} = f(x, u) = \mathbf{v}(x) + \rho u \quad |u| \leq 1 \]

Differential inclusion

\[ \dot{x} \in F(x) = \{ \mathbf{v}(x) + \rho u ; \quad |u| \leq 1 \} \]
2 - Systems with scalar control entering linearly

\[ \dot{x} = f(x) + g(x)u \quad u \in [-1, 1] \]

\[ \dot{x} \in F(x) = \{ f(x) + g(x)u ; \quad u \in [-1, 1] \} \]

**Stabilization in minimum time:** Find a control \( u(\cdot) \) that steers the system to the origin in minimum time.
Example 2

\[ \dot{x} = u \quad u \in [-1, 1] \]

\[
\begin{cases}
\dot{x} = v \\
\dot{v} = u
\end{cases}
\begin{cases}
x(0) = x_0 \\
v(0) = v_0
\end{cases}
\]

Find feedback control \( u = U(x, v) \) steering the system to the origin in minimum time.
Regularity of Multifunctions

The **Hausdorff distance** between two compact sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ is

$$d_H(\Omega_1, \Omega_2) = \min \{ \rho : \Omega_1 \subseteq B(\Omega_2, \rho) \text{ and } \Omega_2 \subseteq B(\Omega_1, \rho) \}$$

$B(\Omega, \rho) \doteq \text{neighborhood of radius } \rho \text{ around the set } \Omega$

![Diagram showing two compact sets and their Hausdorff distance]

$$d_H(\Omega_1, \Omega_2) = \max \{ \rho_1, \rho_2 \}$$

$\rho_1 = \text{maximum distance of points } p \in \Omega_1 \text{ from } \Omega_2$

$\rho_2 = \text{maximum distance of points } p \in \Omega_2 \text{ from } \Omega_1$

A multifunction $x \mapsto G(x) \subset \mathbb{R}^n$ is **Lipschitz continuous** if

$$d_H(G(x), G(y)) \leq L |x - y| \quad x, y \in \mathbb{R}^n.$$

If $U \subset \mathbb{R}^m$ is compact and $f$ is Lipschitz continuous, then the multifunction

$$x \mapsto G(x) \doteq \{ f(x, u) ; \ u \in U \}$$

is Lipschitz continuous.
Equivalence with Differential Inclusions

Control System: \[ \dot{x} = f(x, u) \quad u \in U \] (1)

Differential Inclusion: \[ \dot{x} \in G(x) \] (2)

Theorem 1 (A. Filippov). If \( f \) is continuous and \( U \) is compact, the set of (Caratheodory) trajectories of (1) coincides with the set of trajectories of (2), with \( G(x) \doteq \{ f(x, u) \ ; \ u \in U \} \).

If \( \dot{x}(t) \in G(x(t)) \) a.e., for each fixed \( t \) there exists \( u(t) \in U \) such that \( \dot{x}(t) = f(x(t), u(t)) \). The map \( t \mapsto u(t) \) can be chosen to be measurable.

Theorem 2 (A. Ornelas). Let \( x \mapsto G(x) \) be a bounded, Lipschitz continuous multifunction with convex, compact values. Then there exists a Lipschitz continuous function \( f : \mathbb{R}^n \times U \mapsto \mathbb{R}^n \) such that \( G(x) \doteq \{ f(x, u) \ ; \ u \in U \} \) for all \( x \). Here the \( U \) (the set of control values) is the closed unit ball in \( \mathbb{R}^n \).
Dynamics of a Control System

- Describe the set of all trajectories of the control system

\[ \dot{x} = f(x, u) \quad u \in U, \quad x(0) = x_0 \]  

(1)

- Describe the **Reachable set at time T**:

\[ R(T) = \{ x(T) ; \ x(\cdot) \text{ is a trajectory of } (1) \} \]

Ideal case: an explicit formula for \( R(T) \)

More realistic goal: derive properties of the reachable set

- A priori bounds: \( R(T) \subset B(x_0, \rho) \)

- Is \( R(T) \) closed, connected, with non-empty interior?

- (Local controllability) Is \( R(T) \) a neighborhood of \( x_0 \), for all \( T > 0 \)?

- (Global controllability) Does every point \( x \in \mathbb{R}^n \) lie in the reachable set \( R(T) \), for \( T \) suitably large?
Linear Systems

\[ \dot{x} = Ax + Bu, \quad x(0) = 0 \quad (4) \]

\( A \) is a \( n \times n \) matrix, \( B \) is \( n \times m \), \( u \in \mathbb{R}^m \).

\[ x(T) = \int_0^T e^{(T-s)A} Bu(s) \, ds \quad (5) \]

\[ e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \]

**Theorem 3.** For every \( T > 0 \), the reachable set is

\[ R(T) = \text{span}\{B, AB, A^2B, \ldots, A^{n-1}B\} \]

**Proof.** \( R(T) \) is clearly a vector space

\[ \text{range} \left( e^{tA} \right) \subseteq \text{span}\{I, A, A^2, \ldots\} = \text{span}\{I, A, A^2, \ldots A^{n-1}\} \]

Hence, by (5), \( R(T) \subseteq \text{span}\{B, AB, A^2B, \ldots, A^{n-1}B\} \).

Viceversa, \( p \in R(T)^\perp \) implies

\[ \langle p , \frac{dx^k(t)}{dt} \rangle = 0 \quad \text{for all } k, t \geq 0. \]

At time \( t = 0 \), taking a constant control \( u(t) \equiv \omega \), one finds

\[ \langle p , A^k B \omega \rangle = 0 \quad \text{for all } k \geq 0, \quad \omega \in \mathbb{R}^m \]

\[ p \in \text{span}\{B, AB, A^2B, \ldots A^{n-1}B\}^\perp \]

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Connectedness

The reachable set $R(T)$ is always connected

\[ x_1 = x(T, u_1), \quad x_2 = x(T, u_2) \]

\[ u^\theta(t) = \begin{cases} u_1(t) & \text{if } t \in [0, \theta T] \\ u_2(t) & \text{if } t \in [\theta T, T] \end{cases} \]

$\theta \mapsto x(T, u^\theta)$ is a path inside $R(T)$, connecting $x_1$ with $x_2$
Closure

Example 3. The set of trajectories may not be closed.

\[ \dot{x} = u \quad u \in \{-1, 1\}, \quad x(0) = 0 \]

For each \( k \geq 1 \), define the control \( u : [0, T] \mapsto \{-1, 1\} \)

\[ u_n(t) = \begin{cases} 1 & \text{if } t \in \left[\frac{2jT}{2n}, \frac{(2j + 1)T}{2n}\right] \\ -1 & \text{if } t \in \left[\frac{(2j - 1)T}{2n}, \frac{2jT}{2n}\right] \end{cases} \quad (6) \]

The trajectories \( t \mapsto x_n(t) \) satisfy

\[ x_n(t) \rightarrow x_\infty(t) \equiv 0 \quad \text{uniformly on } [0, T] \]

but \( x_\infty(t) \equiv 0 \) is not a trajectory of the system.
Example 4. The reachable set $R(T) \subset \mathbb{R}^2$ may not be closed.

$$(\dot{x}_1, \dot{x}_2) = (u, x_1^2) \quad u \in \{-1, 1\}, \quad (x_1, x_2)(0) = (0, 0) \quad (7)$$

Choosing the controls $u_n(\cdot)$ in (6), at time $T$ we reach the points

$$P_n = (0, T^3/12n^2)$$

$P_n \to (0, 0)$ as $n \to \infty$, but $(0, 0) \notin R(T)$. Indeed, for any trajectory of the system

$$x_2(T) = \int_0^T x_1^2(s) \, ds = 0$$

only if $x_1(t) \equiv 0$ for all $t$, hence $\dot{x}_1(t) = u(t) = 0$ for almost every $t$. Against the assumption $u(t) \in \{-1, 1\}$
**Theorem 4 (A. Filippov).** Let $x \mapsto G(x)$ be a Lipschitz continuous multifunction with compact, convex values. Then, for every $T \geq 0$ the set of solutions of

$$
\dot{x} \in G(x), \quad x(0) = x_0 \quad t \in [0, T]
$$

is closed w.r.t. uniform convergence.

**Corollary.** Under the previous assumptions, the set of trajectories is a compact subset of $C([0, T])$. Moreover, the reachable set $R(T)$ is compact.

**Proof.** Consider a sequence of trajectories

$$
\dot{x}_n(t) \in G(x_n(t)) \quad t \in [0, T]
$$

$$
x_n(t) \to x(t) \quad \text{uniformly on } [0, T]
$$

Clearly $x(\cdot)$ is Lipschitz continuous, hence differentiable a.e.
Assume $\dot{x}(t) \notin G(x(t))$ at some time $t$. Then $\dot{x}(t)$ is separated by a hyperplane from the convex set $G(x(t))$

$$\langle p, \dot{x}(t) \rangle \geq 3\delta + \max_{\omega \in G(x(t))} \langle p, \omega \rangle$$

$$\langle p, \frac{x(t+\varepsilon t) - x(t)}{\varepsilon} \rangle \geq 2\delta + \max_{\omega \in G(x(t))} \langle p, \omega \rangle$$  (8)

On the other hand, by continuity of $G$,

$$\max_{\omega \in G(y)} \langle p, \omega \rangle \leq \delta + \max_{\omega \in G(x(t))} \langle p, \omega \rangle$$

for $|y - x(t)| \leq \rho$ small. Hence

$$\langle p, \frac{x_n(t+\varepsilon t) - x_n(t)}{\varepsilon} \rangle \leq \max_{\omega \in G(y), |y-x(t)| \leq \rho} \langle p, \omega \rangle \leq \delta + \max_{\omega \in G(x(t))} \langle p, \omega \rangle$$

Letting $n \to \infty$, we obtain

$$\langle p, \frac{x(t+\varepsilon t) - x(t)}{\varepsilon} \rangle \leq \delta + \max_{\omega \in G(x(t))} \langle p, \omega \rangle$$  (9)

in contradiction with (8).
Convex Closure and Extreme Points

Let $\Omega \subset \mathbb{R}^n$ be any compact set. The **convex closure** of $\Omega$, written $\overline{\text{co}}\Omega$, is the smallest closed convex set which contains $\Omega$. It admits the representation

$$\overline{\text{co}}\Omega = \left\{ \sum_{i=0}^{n} \theta_i \omega_i \mid \omega_i \in \Omega, \ \theta_i \geq 0, \ \sum \theta_i = 1 \right\}$$

Hence, if $\Omega \subset \mathbb{R}^n$, the convex closure $\overline{\text{co}}\Omega$ coincides with the set of all convex combinations of $n+1$ elements of $\Omega$.

A point $\omega \in \Omega$ is an **extreme point** if it CANNOT be written as a convex combination

$$\omega = \sum_{i=1}^{k} \theta_i \omega_i, \quad \theta_i \in [0,1], \ \sum \theta_i = 1, \ \omega_i \in \Omega, \ \omega_i \neq \omega$$

The set of extreme points is written as $\text{ext} \Omega$.
Density Theorems

Let $x \mapsto G(x)$ be a multifunction with compact values. We compare the solutions of the differential inclusions

\begin{align*}
\dot{x} & \in \text{ext } G(x) \\
\dot{x} & \in G(x) \\
\dot{x} & \in \overline{co} G(x)
\end{align*}

**Theorem 5 (Relaxation).** Let the multifunction $x \mapsto G(x)$ be *Lipschitz continuous* with compact values. Then set of trajectories of (10) is dense on the set of trajectories of (12)
Bang-bang Property

Assume that, for every solution of

\[ \dot{x} \in \overline{co} G(x) \quad t \in [0, T], \quad x(0) = x_0 \]

one can find a solution of

\[ \dot{y} \in \text{ext} G(x) \quad t \in [0, T], \quad x(0) = x_0 \]

such that \( y(T) = x(T) \). Then we say that the multifunction \( G \) has the \textbf{bang-bang property}.

\[ x_0 \]
\[ \bullet \]
\[ y \]
\[ \bullet \]
\[ x(T)=y(T) \]

**Theorem 6 (Bang-Bang for Linear Systems).** Let \( U \subset \mathbb{R}^m \) be compact. Then, for every solution of

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad u(t) \in U, \quad t \in [0, T] \]

there exists a solution of

\[ \dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = x_0, \quad u(t) \in \text{ext} U, \quad t \in [0, T] \]

such that \( y(T) = x(T) \)

In general, the bang-bang property does not hold for nonlinear systems (Example 4)
A Lyapunov-type Convexity Theorem

\[ f_1, \ldots, f_N \in L^1([0, T]; \mathbb{R}^n) \]

consider a pointwise convex combination of the \( f_i \)

\[
\tilde{f}(t) = \sum_{i=1}^{N} \theta_i(t) f_i(t)
\]

\[ \theta_1, \ldots, \theta_N : [0, T] \mapsto [0, 1], \quad \sum \theta_i(t) = 1 \text{ for all } t \]

\[
\int_0^T \tilde{f}(t) \, dt = \sum_{i=1}^{N} \int_{J_i} f_i(t) \, dt
\]

**Theorem 7.** There exists a partition

\[ [0, T] = \bigcup_{i=1}^{N} J_i, \quad J_i \cap J_k = 0 \text{ if } i \neq k \]
**Proof of Theorem 7.** Consider the set of coefficients of convex combinations

$$\Gamma \doteq \left\{ (w_1, \ldots, w_N) ; \ w_i(t) \in [0,1], \ \sum_{i=1}^{N} w_i(t) = 1, \ \int_0^T \sum_i w_i(t) f_i(t) \, dt = \int_0^T \tilde{f}(t) \, dt \right\}$$

1. $\Gamma$ is non-empty, because $(\theta_1, \ldots, \theta_N) \in \Gamma$
2. $\Gamma \subset L^\infty$ is closed and convex, hence compact in a weak topology
3. By the Krein-Milman Theorem, $\Gamma$ has an extreme point $(w_1^*, \ldots, w_n^*)$
4. By extremality, the functions $w_i^* : [0,T] \mapsto [0,1]$ actually take values in $\{0,1\}$
5. Take $J_i = \{ t \in [0,T] ; \ w_i^*(t) = 1 \}$

**Proof of Theorem 6.**

Consider any control $u : [0,T] \mapsto U \subset \mathbb{R}^m$. Then

$$u(t) = \sum_{i=0}^{m} \theta_i(t) u_i(t) \quad u_i(t) \in \text{ext } U$$

Using the control $u(\cdot)$, the terminal point is

$$x(T) = \int_0^T e^{(T-s)A} Bu(s) \, ds = \int_0^T \sum_{i=0}^{m} \theta_i(t) e^{(T-s)A} Bu_i(s) \, ds$$

Apply previous theorem with $f_i(s) = e^{(T-s)A} Bu_i(s)$. For a suitable partition $[0,T] = J_0 \cup J_1 \cup \cdots \cup J_m$, one has

$$x(T) = \sum_{i=0}^{m} \int_{J_i} e^{(T-s)A} Bu_i(s) \, ds$$

Hence the control

$$u^*(t) = u_i(t) \quad \text{if } t \in J_i$$

takes values in ext $U$ and reaches exactly the same terminal point $x(T)$. 

Baire Category Approach

Baire Category Theorem. Let $K$ be a complete metric space, $(K_n)_{n \geq 1}$ a sequence of open, dense subsets. Then $\bigcap_{n \geq 1} K_n$ is non-empty and dense in $K$.

Alternative proof of the Bang-Bang Theorem

We can assume that all sets $G(x)$ are convex.
Assume there exists at least one trajectory of
\[ \dot{x}(t) \in G(x(t)), \quad x(0) = x_0, \quad x(T) = x_1 \]

The set $K$ of all such trajectories is then a non-empty, compact subset of $C([0,T])$. In particular, $K$ is a complete metric space.

Under suitable assumptions on the multifunction $G$, the set $K^{ext}$ of solutions of
\[ \dot{x} \in \text{ext } G(x), \quad x(0) = x_0, \quad x(T) = x_1 \]
can be written as the intersection of countably many open dense subsets $K_n \subset K$.
Hence by the Baire Cathegory Theorem, $K^{ext} \neq \emptyset$

This technique applies to a class of “concave” multifunctions $G$, introduced in (Bressan-Piccoli, J.Diff.Equat. 1995)
1. For each compact, convex set $\Omega \subset \mathbb{R}^n$, define the function $\varphi_{\Omega} : \Omega \mapsto [0, \infty]$ as

$$
\varphi_{\Omega}(p) = \sup \left\{ \left( \int_0^1 |X(s) - p|^2 \, ds \right)^{1/2} ; \ X : [0, 1] \mapsto \Omega, \ \int_0^1 X(s) \, ds = p \right\}
$$

Observe that $\varphi_{\Omega}^2(p)$ is the maximum variance among all random variables $X$ taking values inside $\Omega$, whose expected value is $E[X] = p$.

2. Each function $\varphi_{\Omega}$ is $\geq 0$, concave, continuous and $\varphi_{\Omega}(p) = 0$ iff $p \in \text{ext} \ \Omega$.

3. The functional

$$
\Phi(x(\cdot)) = \int_0^T \varphi_{G(x)}(\dot{x}(t)) \, dt
$$

is well defined for all trajectories of $\dot{x} \in G(x)$

Moreover $\Phi(x(\cdot)) = 0$ iff $\dot{x}(t) \in \text{ext} \ G(x(t))$ for a.e. $t \in [0, T]$.

4. The sets $K_n = \{ x(\cdot) \in K ; \ \Phi(x) < 1/n \}$ are open and dense in $K$.

5. By the Baire Cathegory Theorem, $K^{ext} = \bigcap_{n \geq 1} K_n \neq \emptyset$.

This shows that, in a topological sense, "almost all" solutions of $\dot{x} \in G(x)$ are actually solutions of $\dot{x} \in \text{ext} \ G(x)$.
Chattering Controls

Given a control system on $\mathbb{R}^n$

$$\dot{x} = f(x, u) \quad u \in U$$

(1)

for each $x$, the set of velocities

$$G(x) = \{ f(x, u) ; \ u \in U \}$$

can not be convex. We seek a new control system

$$\dot{x} = \tilde{f}(x, \tilde{u}) \quad \tilde{u} \in \tilde{U}$$

(13)

such that

$$\tilde{G}(x) = \{ f(x, \tilde{u}) ; \ \tilde{u} \in \tilde{U} \} = \bar{\text{co}} G(x)$$

Since every $\omega \in \bar{\text{co}} G(x)$ is a convex combination of $n + 1$ vectors in $G(x)$, the system (13) can be defined as follows:

$\Delta \doteq \left\{ (\theta_0, \theta_1, \ldots, \theta_n) ; \ \theta_i \in [0, 1], \ \sum_{i=0}^{n} \theta_i = 1 \right\}$

$$\tilde{U} \doteq U \times \cdots \times U \times \Delta$$

(\text{\text{n+1 times})}

$$\tilde{f}(x, \tilde{u}) = \tilde{f}(x, u_1, \ldots, u_n, \theta_0, \ldots, \theta_n) = \sum_{i=0}^{n} \theta_i f(x, u_i)$$

(14)

The system (14) is called a chattering system.

A control $\tilde{u} = (u_0, \ldots, u_n, \theta_0, \ldots, \theta_n) \in \tilde{U}$ is called a chattering control

- The set of trajectories of a chattering system is always closed in $\mathcal{C}([0, T])$
- The reachable set $\tilde{R}(T)$ is compact.
- Every trajectory of (13) can be uniformly approximated by trajectories of the original system (1)
Lie Brackets

Given two smooth vector fields \( f, g \), their **Lie bracket** is defined as

\[
[f, g] \equiv (Dg) \cdot f - (Df) \cdot g
\]

In other words, \([f, g]\) is the directional derivative of \( g \) in the direction of \( f \) minus the directional derivative of \( f \) in the direction of \( g \).

Exponential notation for the flow map: 
\( \tau \mapsto (\exp \tau f)(x) \) denotes the solution of the Cauchy problem

\[
\frac{dw}{d\tau} = f(w), \quad w(0) = x
\]

The Lie bracket can be equivalently characterized as

\[
[f, g](x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \bigg[ (\exp(-\varepsilon g))(\exp(-\varepsilon f))((\exp \varepsilon g)(\exp \varepsilon f)(x) - x) \bigg]
\]

The **Lie algebra** generated by the vector fields \( f_1, \ldots, f_n \) is the space of vector fields generated by the \( f_i \) and by all their iterated Lie brackets

\[
[f_i, f_j], \quad [f_i, [f_j, f_k]], \quad \ldots
\]
Local Controllability

\[ \dot{x} = \sum_{i=1}^{m} f_i(x) u_i \quad u_i \in [-1, 1] \quad (15) \]
\[ x(0) = x_0 \in \mathbb{R}^n \]

**Theorem 8.** If the linear span of all iterated Lie brackets of \( f_1, \ldots, f_n \) at \( x_0 \) is the whole space \( \mathbb{R}^n \), then the system (15) is locally controllable. That means: for every \( T > 0 \) the reachable set \( R(T) \) is a neighborhood of \( x_0 \).

More difficult case: a drift \( f_0 \) is present.

\[ \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i \quad u_i \in [-1, 1] \quad (16) \]

**Theorem 9 (Sussmann-Jurdjevic).** If the linear span of all iterated Lie brackets of \( f_0, f_1, \ldots, f_n \) at \( x_0 \) is the whole space \( \mathbb{R}^n \), then for every \( T > 0 \) the set of points reachable within time \( \leq T \) has non-empty interior.

Local controllability for (16) is a hard problem!
**Example 5.** The position of a car is described by $P = (x, y, \theta)$. The first two variables $x, y$ determine the location of the baricenter, while $\theta$ determines the orientation. If the car advances with unit speed steering to the right, its motion satisfies

$$\frac{dP}{dt} = f(P) = (\cos \theta, \sin \theta, -1).$$

On the other hand, if the car steers to the left, its motion satisfies

$$\frac{dP}{dt} = g(P) = (\cos \theta, \sin \theta, 1).$$

In a typical parking problem, one needs to shift the car toward one side, without changing its orientation. This is obtained by the maneuver:

1. right-forward,
2. left-forward,
3. right-backward,
4. left-backward.

Indeed, the sequence of these four actions generates the Lie bracket

$$[f, g] = (Dg) \cdot f - (Df) \cdot g = 2(\sin \theta, -\cos \theta, 0)$$

which is precisely the desired motion.
The system on \( \mathbb{R}^3 \)

\[
\dot{P} = f(P) u_1 + g(P) u_2, \quad u_1, u_2 \in [-1, 1]
\]

is locally controllable. Indeed,

\[
\text{span}\{f(P), g(P), [f, g](P)\} = \mathbb{R}^3
\]

because it contains the three vectors \((\cos \theta, \sin \theta, -1)\), \((\cos \theta, \sin \theta, -1)\) and \((\sin \theta, -\cos \theta, 0)\)

**Example 6.** The system on \( \mathbb{R}^2 \)

\[
\begin{cases}
\dot{x}_1 = u \\
\dot{x}_2 = x_1^2
\end{cases} \quad u \in [-1, 1] \tag{17}
\]

\((x_1, x_2)(0) = (0, 0)\)

is not locally controllable. Indeed, for every trajectory

\[
x_2(T) = \int_0^T x_1^2(s) \, ds \geq 0
\]

We can write (17) in the form

\[
\dot{x} = f_0(x) + f_1(x) u
\]

\[
f_0 = (0, x_1^2), \quad f_1 = (1, 0)
\]

In this case

\[
[f_1, f_0] = (0, 2x_1), \quad [f_1, [f_1, f_0]] = (0, 2)
\]

hence the vectors \(f_1\) and \([f_1, [f_1, f_0]]\) span \( \mathbb{R}^2 \).

For every \(T > 0\) the reachable set \( R(T) \) has non-empty interior.
Optimal Control Problems

\[ \dot{x} = f(x, u) \quad u \in U, \quad t \in [0, T] \quad (1) \]
\[ x(0) = x_0 \in \mathbb{R}^n \quad (2) \]

Among all trajectories of (1)-(2), we seek one which is optimal w.r.t. some cost criterion:

**Mayer Problem**

\[
\text{minimize} \quad J = \psi(x(T)) \quad (18)
\]

**Lagrange Problem**

\[
\text{minimize} \quad J = \int_{0}^{T} \varphi(t, x(t), u(t)) \, dt \quad (19)
\]

**Bolza Problem**

\[
\text{minimize} \quad J = \int_{0}^{T} \varphi(t, x(t), u(t)) \, dt + \psi(x(T)) \quad (20)
\]

\[ \varphi(x, u) = \text{running cost per unit time} \]
\[ \psi(x) = \text{terminal cost} \]
The minimization is always performed among all control functions $u : [0, T] \mapsto U$

One may also add a terminal constraint, requiring

$$x(T) \in S$$

for some set $S \subset \mathbb{R}^n$

**Basic assumptions:** The cost functions $\varphi, \psi$ are continuous. The set $S$ is closed.

**Equivalence of various formulations**

1. A Mayer Problem can be written as a Lagrange Problem, taking

$$\varphi(x, u) \doteq \nabla \psi(x) \cdot f(x, u)$$

We are then minimizing

$$\int_0^T \nabla \psi(x(t)) \cdot f(x(t), u(t)) \, dt = \int_0^T \nabla \psi(x(t)) \cdot \dot{x}(t) \, dt$$

$$= \int_0^T \left( \frac{d}{dt} \psi(x(t)) \right) \, dt = \psi(x(T)) - \psi(x_0)$$

2. A Lagrange Problem can be written as a Mayer Problem, introducing a new variable $x_{n+1}$ with

$$x_{n+1}(0) = 0, \quad \dot{x}_{n+1} = \varphi(t, x(t), u(t))$$

and defining

$$\psi(x) \doteq x_{n+1}$$

This yields the terminal cost

$$\psi(x(T)) = x_{n+1}(T) = \int_0^T \varphi(t, x(t), u(t)) \, dt$$
Relations with the Calculus of Variations

Standard Problem in the Calculus of Variations

\[
\text{minimize } \int_0^T L(t, x(t), \dot{x}(t)) \, dt \tag{22}
\]

among all absolutely continuous functions \( x : [0,T] \mapsto \mathbb{R}^n \) with
\[
x(0) = x_0, \quad x(T) = x_1
\]

Optimal Control Problem

\[
\text{minimize } \int_0^T \varphi(t, x(t), u(t)) \, dt \tag{19}
\]
\[
\dot{x} = f(x, u) \quad u \in U, \quad t \in [0,T] \tag{1}
\]
\[
x(0) = x_0, \quad x(T) = x_1 \tag{2}
\]

1. We can write (22) as an optimal control problem by setting
\[
\dot{x} = u, \quad u \in U \equiv \mathbb{R}^n
\]
\[
\varphi(t, x, u) \equiv L(t, x, u)
\]

2. We can write (19) in the form (22) by setting
\[
L(t, x, p) \equiv \min \{ \varphi(t, x, u) ; \ u \in U, \ f(x, u) = p \}
\]
\[
L(t, x, p) = \infty \quad \text{if } p \notin \{ f(x, u) ; u \in U \}
\]

\( L(t, x, p) \) is the minimum running cost, among all controls that yield the speed \( \dot{x}(t) = p \).
Existence of Optimal Controls

Mayer problem:

\[
\text{minimize} \quad J = \psi(x(T))
\]  \hspace{1cm} (18)

for the system

\[
\dot{x} = f(x,u), \quad u \in U
\]  \hspace{1cm} (1)

with constraints

\[
x(0) = x_0, \quad x(T) \in S
\]  \hspace{1cm} (2)
Theorem 10 (Existence for Mayer Problem).
Assume that, for each $x$, the set of velocities

$$G(x) \doteq \{ f(x,u) ; \ u \in U \}$$

is convex. If there exists at least one trajectory $x(\cdot)$ that satisfies (1)-(2), then the Mayer problem (18) admits an optimal solution.

Proof. By Theorem 4, the reachable set $R(T)$ is compact.

By the assumptions, $R(T) \cap S$ is non-empty and compact.

The continuous function $\psi$ admits a global minimum on the compact set $R(T) \cap S$. This yields the optimal solution.

Bolza Problem:

$$\minimize J = \int_0^T \varphi(t, x(t), u(t)) \, dt + \psi(x(T)) \quad (20)$$

for the system (1) with initial and terminal conditions (2).

Theorem 11 (Existence for the Bolza Problem). Assume that, for every $t, x$, the set

$$G^+(t, x) \doteq \{(y, y_{n+1}) \in \mathbb{R}^{n+1} ; \ y = f(x,u), \ y_{n+1} \geq \varphi(t, x, u) \ \text{for some} \ u \in U \}$$

is convex. If there exists at least one trajectory $x(\cdot)$ that satisfies (1)-(2), then the Bolza problem (20) admits an optimal solution.
**Remark:** For the problem:

\[
\text{minimize } \int_0^T L(t, x(t), u(t)) \, dt \tag{22}
\]

subject to

\[
\dot{x} = u, \quad x(0) = x_0, \quad x(T) = x_1
\]

corresponding to the standard problem in the Calculus of Variations, the set

\[
G^+(t, x) \equiv \{(y, y_{n+1}) \in \mathbb{R}^{n+1}; \quad y = f(x, u), \quad y_{n+1} \geq \varphi(t, x, u) \text{ for some } u \in U\}
\]

\[
= \{(u, y_{n+1}) \in \mathbb{R}^{n+1}; \quad y_{n+1} \geq L(t, x, u)\}
\]

is precisely the epigraph of the function \( u \mapsto L(t, x, u) \).

This is a convex set iff \( L(t, x, \dot{x}) \) is convex as a function of \( \dot{x} \).
First Order Variations

Let \( t \mapsto x(t) \) be a solution to the O.D.E.
\[
\dot{x} = g(t, x) \quad x \in \mathbb{R}^n
\] (24)
Assume \( g \) measurable w.r.t. \( t \) and continuously differentiable w.r.t. \( x \)

First order perturbation:
\[
x_{\varepsilon}(t) = x(t) + \varepsilon v(t) + o(\varepsilon)
\] (25)
\( o(\varepsilon) \) = infinitesimal of higher order w.r.t. \( \varepsilon \)

If \( t \mapsto x_{\varepsilon}(t) \) is another solution of (24), letting \( \varepsilon \to 0 \) we find a linearized evolution equation for the first order tangent vector \( v \)
\[
\dot{v}(t) = A(t)v(t) \quad \quad A(t) = D_xg(t, x(t))
\] (26)

adjoint system: \( \dot{p}(t) = -p(t)A(t) \) (27)

Here \( A \) is an \( n \times n \) matrix, with entries \( A_{ij} = \partial g_i/\partial x_j \), \( p \in \mathbb{R}^n \) is a row vector and \( v \in \mathbb{R}^n \) is a column vector.

If \( t \mapsto p(t) \) and \( t \mapsto v(t) \) satisfy (27) and (26), then the product \( t \mapsto p(t)v(t) \) is constant in time:
\[
\frac{d}{dt}(p(t)v(t)) = \dot{p}(t)v(t) + p(t)\dot{v}(t) = -p(t)A(t)v(t) + p(t)A(t)v(t) = 0
\]
Necessary Conditions for Optimality

\[ \dot{x} = f(x, u) \quad t \in [0, T], \quad x(0) = x_0 \] (28)

Family of admissible control functions: \( \mathcal{U} \doteq \{ u : [0, T] \mapsto U \} \)

For \( u(\cdot) \in \mathcal{U} \), the trajectory of (28) is \( t \mapsto x(t, u) \)

Mayer problem, free terminal point:

\[ \max_{u \in \mathcal{U}} \psi(x(T, u)) \] (29)
Assume: \( t \mapsto u^*(t) \) is an optimal control
\( t \mapsto x^*(t) = x(t, u^*(t)) \) is the corresponding optimal trajectory.

The value \( \psi(x(T, u^*)) \) cannot be increased by any perturbation of the control \( u^*(\cdot) \).

Fix a time \( \tau \in ]0, T] \), \( \omega \in U \) and consider the needle variation \( u_\varepsilon \in U \)

\[
u_\varepsilon(t) = \begin{cases} 
\omega & \text{if} \quad t \in [\tau - \varepsilon, \tau] \\
 u^*(t) & \text{if} \quad t \notin [\tau - \varepsilon, \tau] 
\end{cases}
\]

Call \( t \mapsto x_\varepsilon(t) = x(t, u_\varepsilon) \) the perturbed trajectory.

We shall compute the terminal point \( x_\varepsilon(T) = x(T, u_\varepsilon) \) and check that the value of \( \psi \) is not increased by the perturbation.
Assuming that the optimal control $u^*$ is continuous at time $t = \tau$, we have

$$v(\tau) \doteq \lim_{\epsilon \to 0} \frac{x_\epsilon(\tau) - x^*(\tau)}{\epsilon} = f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)) \quad (31)$$

Indeed, $x_\epsilon(\tau - \epsilon) = x^*(\tau - \epsilon)$ and on the small interval $[\tau - \epsilon, \tau]$ we have

$$\dot{x}_\epsilon \approx f(x^*(\tau), \omega), \quad \dot{x}^* \approx f(x^*(\tau), u^*(\tau)).$$

Since $u_\epsilon = u^*$ on the remaining interval $t \in [\tau, T]$, the evolution of the tangent vector

$$v(t) \doteq \lim_{\epsilon \to 0} \frac{x_\epsilon(t) - x^*(t)}{\epsilon} \quad t \in [\tau, T]$$

is governed by the linear equation

$$\dot{v}(t) = A(t) v(t) \quad A(t) \doteq D_x f(x^*(t), u^*(t)). \quad (32)$$

By maximality, $\psi(x_\epsilon(T)) \leq \psi(x^*(T))$, therefore

$$\nabla \psi(x^*(T)) \cdot v(T) \leq 0. \quad (33)$$
**Summing up:** For every time $\tau$ and every control value $\omega \in U$, we can generate the vector
\[
v(\tau) = f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau))
\]
and propagate it forward in time, by solving the linearized equation (32). The inequality (33) is then a necessary condition for optimality.

Instead of propagating the (infinitely many) vectors $v(\tau)$ forward in time, it is more convenient to propagate the single vector $\nabla \psi$ backward. We thus define the row vector $t \mapsto p(t)$ as the solution of
\[
\dot{p}(t) = -p(t)A(t), \quad p(T) = \nabla \psi(x^*(T))
\]
(34)
This yields $p(t)v(t) = p(T)v(T)$ for every $t$. In particular, (33) implies
\[
p(\tau) \cdot \left[ f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)) \right] = \nabla \psi(x^*(T)) \cdot v(T) \leq 0
\]
\[
p(\tau) \cdot \dot{x}^*(\tau) = p(\tau) \cdot f(x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(x^*(\tau), \omega) \right\}.
\]
For every time $\tau \in ]0, T]$, the speed $\dot{x}^*(\tau)$ corresponding to the optimal control $u^*(\tau)$ is the one having has inner product with $p(\tau)$ as large as possible.
Pontryagin Maximum Principle (Mayer Problem, free terminal point)

Consider the control system
\[ \dot{x} = f(x, u), \quad u(t) \in U, \quad t \in [0, T] \]  
with initial data \[ x(0) = x_0. \]

Let \( t \mapsto u^*(t) \) be an optimal control and \( t \mapsto x^*(t) = x(t, u^*) \) be the optimal trajectory for the maximization problem
\[ \max_{u \in U} \psi(x(T, u)). \]

Define the vector \( t \mapsto p(t) \) as the solution to the linear adjoint system
\[ \dot{p}(t) = -p(t) A(t), \quad A(t) = D_x f(x^*(t), u^*(t)) \]
with terminal condition \[ p(T) = \nabla \psi(x^*(T)). \]

Then, for almost every \( \tau \in [0, T] \) the following maximality condition holds
\[ p(\tau) \cdot f(x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(x^*(\tau), \omega) \right\} \]
Computing the Optimal Control

STEP 1: solve the pointwise maximization problem (39), obtaining the optimal control $u^*$ as a function of $p, x$, i.e.

$$ u^*(x, p) = \arg\max_{\omega \in U} \{ p \cdot f(x, \omega) \} \quad (40) $$

STEP 2: solve the two-point boundary value problem

$$
\begin{align*}
\dot{x} &= f(x, u^*(x, p)) \\
\dot{p} &= -p \cdot D_x f(x, u^*(x, p)) \\
x(0) &= x_0 \\
p(T) &= \nabla \psi(x(T))
\end{align*}
$$

In general, the function $u^* = u^*(p, x)$ in (40) is highly nonlinear. It may be multivalued or discontinuous.

The two-point boundary value problem (41) can be solved by a shooting method: Guess an initial value $p(0) = p_0$ and solve the corresponding Cauchy problem. Try to adjust the value of $p_0$ so that the terminal values $x(T), p(T)$ satisfy the given conditions.
Example 1 (Linear pendulum).

\[ q(t) = \text{position of a linearized pendulum, controlled by an external force with magnitude } u(t) \in [-1, 1]. \]

\[ \ddot{q}(t) + q(t) = u(t), \quad q(0) = \dot{q}(0) = 0, \quad u(t) \in [-1, 1] \]

We wish to maximize the terminal displacement \( q(T) \).

Equivalent control system: \( x_1 = q, x_2 = \dot{q} \)

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u - x_1 \\
\end{aligned}
\]

\[
\begin{aligned}
 x_1(0) &= 0 \\
 x_2(0) &= 0 \\
\end{aligned}
\]

maximize \( x_1(T) \) over all controls \( u : [0, T] \mapsto [-1, 1] \)

Let \( t \mapsto x^*(t) = x(t, u^*) \) be an optimal trajectory. The linearized equation for a tangent vector is

\[
\begin{pmatrix}
\dot{v}_1 \\
\dot{v}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

The corresponding adjoint vector \( p = (p_1, p_2) \) satisfies

\[
(p_1, p_2) = -(p_1, p_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (p_1, p_2)(T) = \nabla \psi(x^*(T)) = (1, 0) \tag{42}
\]

because \( \psi(x) = x_1. \)

In this special linear case, we can explicitly solve (42) without needing to know \( x^*, u^* \). An easy computation yields

\[
(p_1, p_2)(t) = \begin{pmatrix} \cos(T - t), \sin(T - t) \end{pmatrix} \tag{43}
\]
For each $t$, we must now choose the value $u^*(t) \in [-1, 1]$ so that

$$p_1 x_2 + p_2(-x_1 + u^*) = \max_{\omega \in [-1, 1]} p_1 x_2 + p_2(-x_1 + \omega)$$

By (43), the optimal control is

$$u^*(t) = \text{sign}(p_2(t)) = \text{sign}(\sin(T - t))$$
**Example 2.** Consider the problem on $\mathbb{R}^3$

\[
\text{maximize } x_3(T) \quad \text{over all controls } u : [0, T] \mapsto [-1, 1]
\]

for the system

\[
\begin{align*}
\dot{x}_1 &= u \\
\dot{x}_2 &= -x_1 \\
\dot{x}_3 &= x_2 - x_1^2
\end{align*}
\]

\[
\begin{align*}
x_1(0) &= 0 \\
x_2(0) &= 0 \\
x_3(0) &= 0
\end{align*}
\]

The adjoint equations take the form

\[
(p_1, p_2, p_3) = (p_2 + 2x_1 p_3, -p_3, 0) \quad (p_1, p_2, p_3)(T) = (0, 0, 1) \quad (44)
\]

Maximixing the inner product $p \cdot \dot{x}$ we obtain the optimality conditions for the control $u^*$

\[
p_1 u^* + p_2 (-x_1) + p_3 (x_2 - x_1^2) = \max_{\omega \in [-1, 1]} p_1 \omega + p_2 (-x_1) + p_3 (x_2 - x_1^2) \quad (45)
\]

\[
\begin{align*}
\begin{cases}
    u^* = 1 & \text{if } p_1 > 0 \\
    u^* \in [-1, 1] & \text{if } p_1 = 0 \\
    u^* = -1 & \text{if } p_1 < 0
\end{cases}
\end{align*}
\]

Solving the terminal value problem (44) for $p_2, p_3$ we find

\[
p_3(t) \equiv 1, \quad p_2(t) = T - t
\]
The function $p_1$ can now be found from the equations
\[ \ddot{p}_1 = -1 + 2u^* = -1 + 2\text{sign}(p_1), \quad p_1(T) = 0, \quad \dot{p}_1(0) = p_2(0) = T \]
with the convention: $\text{sign}(0) = [-1, 1]$. The only solution is found to be
\[
p_1(t) = \begin{cases} 
-\frac{3}{2} \left( \frac{T}{3} - t \right)^2 & \text{if } 0 \leq t \leq T/3 \\
0 & \text{if } T/3 \leq t \leq T 
\end{cases}
\]
The optimal control is
\[
u^*(t) = \begin{cases} 
-1 & \text{if } 0 \leq t \leq T/3 \\
1/2 & \text{if } T/3 \leq t \leq T 
\end{cases}
\]
Observe that on the interval $[T/3, T]$ the optimal control is derived not from the maximality condition (45) but from the equation $\ddot{p}_1 = (-1 + 2u) \equiv 0$. An optimal control with this property is called singular.
Tangent Cones

\[ \dot{x} = f(x, u) \quad u \in U, \quad x(0) = x_0 \]

Let \( t \mapsto x^*(t) = x(t, u^*) \) be a reference trajectory. Given \( \tau \in ]0, T], \omega \in U \), consider the family of needle variations

\[ u_\varepsilon(t) = \begin{cases} \omega & \text{if } t \in [\tau - \varepsilon, \tau] \\ u^*(t) & \text{if } t \notin [\tau - \varepsilon, \tau] \end{cases} \]

(30)

Call \( v^{\tau, \omega}(T) = \lim_{\varepsilon \to 0} \frac{x(T, u_\varepsilon) - x(T, u^*)}{\varepsilon} \) the first order variation of the terminal point of the corresponding trajectory.

Define \( \Gamma \) as the smallest convex cone containing all vectors \( v^{\tau, \omega} \). This is a cone of feasible directions, i.e. directions in which we can move the terminal point \( x(T, u^*) \) by suitably perturbing the control \( u^* \).
Consider a terminal constraint \( x(T) \in S \), where

\[
S \doteq \{ x \in \mathbb{R}^n ; \ \phi_i(x) = 0, \quad i = 1, \ldots, N \}.
\]

Assume that the \( N+1 \) gradients \( \nabla \psi, \nabla \phi_1, \ldots, \nabla \phi_N \) are linearly independent at the point \( x^* \). Then the tangent space to \( S \) at \( x^* \) is

\[
T_S = \{ v \in \mathbb{R}^n ; \ \nabla \phi_i(x^*) \cdot v = 0 \quad i = 1, \ldots, N \}.
\]

The tangent cone to the set

\[
S^+ = \{ x \in S ; \ \psi(x) \geq \psi(x^*) \}
\]
is

\[
T_{S^+} = \{ v \in \mathbb{R}^n ; \ \nabla \psi(x^*) \cdot v \geq 0, \quad \nabla \phi_i(x^*) \cdot v = 0 \quad i = 1, \ldots, N \}
\]

When \( x^* = x^*(T) \), we think of \( T_{S^+} \) as the cone of profitable directions, i.e. those directions in which we would like to move the terminal point, in order to increase the value of \( \psi \) and still satisfy the constraint \( x(T) \in S \).

**Lemma 1.** A vector \( p \in \mathbb{R}^n \) satisfies

\[
p \cdot v \geq 0 \quad \text{for all } v \in T_{S^+}
\]

if and only if it can be written as a linear combination

\[
p = \lambda_0 \nabla \psi(x^*) + \sum_{i=1}^{N} \lambda_i \nabla \phi_i(x^*) \quad \text{with } \lambda_0 \geq 0.
\]
Mayer Problem with terminal constraints

\[ \max \psi(x(T)) \]

for the control system

\[ \dot{x} = f(x,u) \quad u(t) \in U \quad t \in [0,T] \]

with initial and terminal constraints

\[ x(0) = x_0, \quad \phi_i(x(T)) = 0, \quad i = 1, \ldots, N \]

**PMP, geometric version.** Let \( t \mapsto x^*(t) = x(t,u^*) \) be an optimal trajectory, corresponding to the control \( u^*(\cdot) \). Then the cones \( \Gamma \) and \( T_{S^+} \) are weakly separated, i.e. there exists a non-zero vector \( p(T) \) such that

\[ p(T) \cdot v \geq 0 \quad \text{for all} \quad v \in T_{S^+} \quad (48) \]

\[ p(T) \cdot v \leq 0 \quad \text{for all} \quad v \in \Gamma \quad (49) \]
PMP, analytic version. Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory, corresponding to the control $u^*(\cdot)$. Then there exists a non-zero vector function $t \mapsto p(t)$ such that

$$p(T) = \lambda_0 \nabla \psi(x^*(T)) + \sum_{i=1}^{N} \lambda_i \nabla \phi_i(x^*(T)) \quad \text{with} \quad \lambda_0 \geq 0 \quad (50)$$

$$\dot{p}(t) = -p(t) D_x f(x^*(t), u^*(t)) \quad t \in [0, T] \quad (51)$$

$$p(\tau) \cdot f(x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(x^*(\tau), \omega) \right\} \quad \text{for a.e.} \, \tau \in [0, T]. \quad (52)$$

Indeed, Lemma 1 states that \( 48 \iff (50) \)

We now show that \( 49 \iff (51)+(52) \).

Recall that every tangent vector $v^{\tau,\omega}$ satisfies the linear evolution equation

$$\dot{v}^{\tau,\omega}(t) = D_x f(x^*(t), u^*(t)) \, v^{\tau,\omega}(t)$$

If $t \mapsto p(t)$ satisfies (51), then the product $p(t) \cdot v^{\tau,\omega}(t)$ is constant. Hence

$$p(T) \cdot v^{\tau,\omega}(T) \leq 0$$

if and only if

$$p(\tau) \cdot v^{\tau,\omega}(\tau) \leq 0$$

if and only if

$$p(\tau) \cdot \left[ f(x^*(\tau), \omega) - f(x^*(\tau), u^*(\tau)) \right] \leq 0$$

if and only if (52) holds.
High Order Conditions?

The Pontryagin Maximum Principle is a first order necessary condition.
For the Mayer problem
\[
\text{maximize } \psi(x(T,u))
\]
if \(u^*(\cdot)\) is a given control, and if we can find a sequence of perturbed controls \(u_\varepsilon\) such that
\[
\lim_{\varepsilon \to 0} \frac{\psi(x(T,u_\varepsilon)) - \psi(x(T,u^*))}{\|u_\varepsilon - u^*\|_{L^1}} > 0
\]
then the PMP will detect the non-optimality of \(u^*\). However, if the increment in the value of \(\psi\) is of second or higher order w.r.t. the variation \(\|u_\varepsilon - u^*\|_{L^1}\), then the conclusion of the PMP may still hold, even if \(u^*\) is not optimal.

Example 3.
\[
\text{maximize } \psi(x(T)) = x_2(T)
\]
for the system
\[
(\dot{x}_1, \dot{x}_2) = (u, x_1^2), \quad (x_1, x_2)(0) = (0, 0), \quad u(t) \in [-1, 1].
\]
The constant control \(u^*(t) \equiv 0\) yields the constant trajectory \((x_1^*, x_2^*)(t) \equiv (0, 0)\). The corresponding adjoint vector satisfying
\[
(\dot{p}_1, \dot{p}_2) = (-2x_1p_2, 0) = (0, 0), \quad (p_1, p_2)(T) = (0, 1)
\]
is trivially found to be \((p_1, p_2)(t) \equiv (0, 1)\). Hence the maximality condition
\[
p_1u^* + p_2(x_1^*)^2 = 0 = \max_{\omega \in [-1, 1]} 0 \cdot \omega + 1 \cdot 0
\]
is satisfied for every \(t \in [0, T]\). However, the control \(u^* \equiv 0\) produces the worst possible outcome. Any other control \(u \neq u^*\) would yield
\[
x_2(T, u) = \int_0^T \left( \int_0^t u(s) \, ds \right)^2 \, dt > x_2(T, u^*) = 0.
\]
Notice that in this case the increase in \(\psi\) is only of second order w.r.t. the perturbation \(u - u^*\)
\[
x_2(T, u) \leq \int_0^T \left( \int_0^T |u(s)| \, ds \right)^2 \, dt \leq T \cdot \|u\|_{L^1}^2
\]
Lagrange Minimization Problem, fixed terminal point

\[
\text{minimize} \quad \int_0^T L(t, x, u) \, dt \quad (53)
\]

for the control system on \(\mathbb{R}^n\)

\[
\dot{x} = f(t, x, u), \quad u(t) \in U \quad (54)
\]

with initial and terminal constraints

\[
x(0) = x_0, \quad x(T) = x^\# \quad (55)
\]

PMP, Lagrange problem.

Let \(t \mapsto x^*(t) = x(t, u^*)\) be an optimal trajectory, corresponding to the optimal control \(u^*(\cdot)\). Then there exist a constant \(\lambda_0 \geq 0\) and a row vector \(t \mapsto p(t)\) (not both = 0) such that

\[
\dot{p}(t) = -p(t) D_x f(t, x^*(t), u^*(t)) - \lambda_0 D_x L(t, x^*(t), u^*(t)) \quad (56)
\]

\[
p(t) \cdot f(t, x^*(t), u^*(t)) + \lambda_0 L(t, x^*(t), u^*(t)) = \min_{\omega \in U} \left\{ p(t) \cdot f(t, x^*(t), \omega) + \lambda_0 L(t, x^*(t), \omega) \right\}. \quad (57)
\]

This follows by applying the previous results to the Mayer problem

\[
\text{minimize} \quad x_{n+1}(T)
\]

with

\[
\dot{x}_{n+1} = L(t, x, u), \quad x_{n+1}(0) = 0
\]

Because of the terminal constraints \((x_1, \ldots, x_n)(T) = (x_1^\#, \ldots, x_n^\#)\), the only requirement on the terminal value \((p_1, \ldots, p_n, p_{n+1})(T)\) is

\[
p_{n+1}(T) \geq 0
\]

Observe that \(\dot{p}_{n+1} = 0\), hence \(p_{n+1}(t) \equiv \lambda_0\) for some constant \(\lambda_0 \geq 0\).
Applications to the Calculus of Variations

Standard problem of the Calculus of Variations:

\[
\text{minimize} \quad \int_0^T L(t, x(t), \dot{x}(t)) \, dt \quad (58)
\]

over all absolutely continuous functions \(x : [0, T] \mapsto \mathbb{R}^n\) such that

\[
x(0) = x_0, \quad x(T) = x^\# \quad (55)
\]

This corresponds to the optimal control problem (53), for the control system

\[
\dot{x} = u, \quad u \in \mathbb{R}^n \quad (59)
\]

We assume that \(L\) is smooth, and that \(x^*(\cdot)\) is an optimal solution. By the Pontryagin Maximum Principle (56)-(57), there exist a constant \(\lambda_0 \geq 0\) and a row vector \(t \mapsto p(t)\) (not both = 0) such that

\[
\dot{p}(t) = -\lambda_0 \frac{\partial}{\partial x} L(t, x^*(t), \dot{x}^*(t)) \quad (60)
\]

\[
p(t) \cdot \dot{x}^*(t) + \lambda_0 L(t, x^*(t), \dot{x}^*(t)) = \min_{\omega \in \mathbb{R}^n} \left\{ p(t) \cdot \omega + \lambda_0 L(t, x^*(t), \omega) \right\} \quad (61)
\]

If \(\lambda_0 = 0\), then \(p(t) \neq 0\). But in this case \(\dot{x}^*\) cannot provide a minimum over the whole space \(\mathbb{R}^n\). This contradiction shows that we must have \(\lambda_0 > 0\).

Since \(\lambda_0, p\) are determined up to a positive scalar multiple, we can assume \(\lambda_0 = 1\). With this choice (61) implies

\[
p(t) = -\frac{\partial}{\partial \dot{x}} L(t, x^*(t), \dot{x}^*(t)) \quad (62)
\]
The evolution equation

\[ \dot{p}(t) = -\frac{\partial}{\partial x} L(t, x^*(t), \dot{x}^*(t)) \]  

(60)

now yields the famous **Euler-Lagrange equations**

\[ \frac{d}{dt} \left[ \frac{\partial}{\partial \dot{x}} L(t, x^*(t), \dot{x}^*(t)) \right] = \frac{\partial}{\partial x} L(t, x^*(t), \dot{x}^*(t)). \]  

(63)

Moreover, the minimality condition

\[ p(t) \cdot \dot{x}^*(t) + L(t, x^*(t), \dot{x}^*(t)) = \min_{\omega \in \mathbb{R}^n} \left\{ p(t) \cdot \omega + L(t, x^*(t), \omega) \right\} \]

(61)

yields the **Weierstrass necessary conditions**

\[ L(t, x^*(t), \omega) \geq L(t, x^*(t), \dot{x}^*(t)) + \frac{\partial L(t, x^*(t), \dot{x}^*(t))}{\partial \dot{x}} \cdot (\omega - \dot{x}^*(t)) \]

(64)

for every \( \omega \in \mathbb{R}^n \).
Viscosity Solutions of Hamilton-Jacobi Equations

One-sided Differentials

The set of super-differentials of $u$ at a point $x$ is

$$D^+ u(x) = \left\{ p \in \mathbb{R}^n ; \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}$$

The set of sub-differentials of $u$ at $x$ is

$$D^- u(x) = \left\{ p \in \mathbb{R}^n ; \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0 \right\}$$
Example 1. Consider the function

\[ u(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\sqrt{x} & \text{if } x \in [0, 1] \\
1 & \text{if } x > 1
\end{cases} \]

In this case we have

\[ D^+ u(0) = \emptyset, \quad D^- u(0) = [0, \infty[ \]

\[ D^+ u(x) = D^- u(x) = \left\{ \frac{1}{2\sqrt{x}} \right\} \quad x \in ]0, 1[ \]

\[ D^+ u(1) = [0, 1/2], \quad D^- u(1) = \emptyset \]
Characterization of super- and sub-differentials

Lemma 1. Let $u \in C(\Omega)$. Then

(i) $p \in D^+u(x)$ if and only if there exists a function $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at $x$.

(ii) $p \in D^-u(x)$ if and only if there exists a function $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local minimum at $x$.

By adding a constant, it is not restrictive to assume that $\varphi(x) = u(x)$. In this case, we are saying that $p \in D^+u(x)$ iff there exists a smooth function $\varphi \geq u$ with $D\varphi(x) = p$, $\varphi(x) = u(x)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Illustration of Lemma 1.}
\end{figure}
Proof. Assume that \( p \in D^+ u(x) \). Then we can find a continuous, increasing function \( \sigma : [0, \infty[ \mapsto \mathbb{R} \), with \( \sigma(0) = 0 \), such that

\[
    u(y) - u(x) - p \cdot (y - x) \leq \sigma(|y - x|)|y - x| = o(1) \cdot (y - x)
\]

We then define

\[
    \rho(r) \doteq \int_0^r \sigma(t) \, dt
\]

\[
    \varphi(y) \doteq u(x) + p \cdot (y - x) + \rho(2|y - x|)
\]

and check that \( \varphi \in C^1 \) and \( u - \varphi \) has a local maximum at \( x \).

Conversely, if \( \varphi \in C^1 \) and \( u - \varphi \) has a local maximum at \( x \), then

\[
    u(y) - u(x) \leq \varphi(y) - \varphi(x)
\]

for every \( y \) in a neighborhood of \( x \), and hence

\[
    \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq \limsup_{y \to x} \frac{\varphi(y) - \varphi(x) - p \cdot (y - x)}{|y - x|} = 0
\]

Remark: By possibly replacing \( \varphi \) with \( \tilde{\varphi}(y) = \varphi(y) \pm |y - x|^2 \), a strict local maximum or local minimum at the point \( x \) can be achieved.
Properties of super- and sub-differentials

Lemma 2. Let \( u \in \mathcal{C}(\Omega) \). Then

(i) If \( u \) is differentiable at \( x \), then
\[
D^+ u(x) = D^- u(x) = \{ \nabla u(x) \}
\]

(ii) If the sets \( D^+ u(x) \) and \( D^- u(x) \) are both non-empty, then \( u \) is differentiable at \( x \).

(iii) The sets of points where a one-sided differential exists:
\[
\Omega^+ = \{ x \in \Omega; \ D^+ u(x) \neq \emptyset \}, \quad \Omega^- = \{ x \in \Omega; \ D^- u(x) \neq \emptyset \}
\]
are both non-empty. Indeed, they are dense in \( \Omega \).

Proof of (i). Assume \( u \) is differentiable at \( x \).

Trivially, \( \nabla u(x) \in D^\pm u(x) \).

On the other hand, if \( \varphi \in \mathcal{C}^1(\Omega) \) is such that \( u - \varphi \) has a local maximum at \( x \), then \( \nabla \varphi(x) = \nabla u(x) \).

Hence \( D^+ u(x) \) cannot contain any vector other than \( \nabla u(x) \).
Proof of (ii). Assume that the sets $D^+u(x)$ and $D^-u(x)$ are both non-empty. Then we can find $\varphi_1, \varphi_2 \in C^1(\Omega)$ such that, for $y$ near $x$,

$$\varphi_1(x) = u(x) = \varphi_2(x), \quad \varphi_1(y) \leq u(y) \leq \varphi_2(y)$$

By comparison, this implies that $u$ is differentiable at $x$ and $\nabla u(x) = \nabla \varphi_1(x) = \nabla \varphi_2(x)$.

Proof of (iii). Fix any ball $B(x_0, \rho) \subset \Omega$.

By choosing $\varepsilon > 0$ sufficiently small, the smooth function

$$\varphi(x) \doteq u(x_0) - \frac{|x - x_0|^2}{2\varepsilon}$$

is strictly negative on the boundary of the ball, where $|x - x_0| = \rho$.

Since $u(x_0) = \varphi(x_0)$, the function $u - \varphi$ attains a local minimum at an interior point $x \in B(x_0, \rho)$.

By Lemma 1, the sub-differential of $u$ at $x$ is non-empty: $\nabla \varphi(x) = (x - x_0)/\varepsilon \in D^-u(x)$.
Viscosity Solutions

\[ F(x, u(x), \nabla u(x)) = 0 \]  \hspace{1cm} (HJ)

**Definition 1.** Let \( F : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) be continuous.

\( u \in C(\Omega) \) is a **viscosity subsolution** of (HJ) if

\[ F(x, u(x), p) \leq 0 \quad \text{for every} \quad x \in \Omega, \; p \in D^+ u(x). \]

\( u \in C(\Omega) \) is a **viscosity supersolution** of (HJ) if

\[ F(x, u(x), p) \geq 0 \quad \text{for every} \quad x \in \Omega, \; p \in D^- u(x). \]

We say that \( u \) is a **viscosity solution** of (HJ) if it is both a supersolution and a subsolution in the viscosity sense.

**Evolution equation:**

\[ u_t + H(t, x, u, \nabla u) = 0. \]  \hspace{1cm} (E)

**Definition 2.** \( u \in C(\Omega) \) is a **viscosity subsolution** of (E) if, for every \( C^1 \) function \( \varphi = \varphi(t, x) \) such that \( u - \varphi \) has a local maximum at \((t, x)\), there holds

\[ \varphi_t(t, x) + H(t, x, u, \nabla \varphi) \leq 0. \]

\( u \in C(\Omega) \) is a **viscosity supersolution** of (E) if, for every \( C^1 \) function \( \varphi = \varphi(t, x) \) such that \( u - \varphi \) has a local minimum at \((t, x)\), there holds

\[ \varphi_t(t, x) + H(t, x, u, \nabla \varphi) \geq 0. \]
• The definition of subsolution imposes conditions on $u$ only on the set $\Omega^+$ of points $x$ where the super-differential $D^+ u(x)$ is non-empty. Even if $u$ is merely continuous, say nowhere differentiable, this set is non-empty and dense in $\Omega$.

• If $u$ is a $C^1$ function that satisfies

$$F(x, u(x), \nabla u(x)) = 0$$ \hspace{1cm} (HJ)

at every $x \in \Omega$, then $u$ is also a solution in the viscosity sense.

• Viceversa, if $u$ is a viscosity solution, then the equality (HJ) must hold at every point $x$ where $u$ is differentiable. If $u$ is Lipschitz continuous, then by Rademacher’s theorem it is a.e. differentiable. Hence (HJ) holds a.e. in $\Omega$.

**Example 2.** The function $u(x) = |x|$ is a viscosity solution of

$$F(x, u, u_x) := 1 - |u_x| = 0$$ \hspace{1cm} (1)

defined on the whole real line. Indeed, $u$ is differentiable and satisfies the equation at all points $x \neq 0$. Moreover, we have

$$D^+ u(0) = \emptyset, \quad D^- u(0) = [-1, 1].$$

To show that $u$ is a subsolution, there is nothing else to check. To show that $u$ is a supersolution, take any $p \in [-1, 1]$. Then $1 - |p| \geq 0$, as required.

Notice that $u(x) = |x|$ is NOT a viscosity solution of the equation

$$|u_x| - 1 = 0$$ \hspace{1cm} (2)

Indeed, at $x = 0$, taking $p = 0 \in D^- u(0)$ we find $|0| - 1 < 0$. Therefore, $u(x) = |x|$ is a viscosity subsolution of (2), but not a supersolution.
Lemma 3. Let \( u_\varepsilon \) be a sequence of smooth solutions to the viscous equation

\[
F(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) = \varepsilon \Delta u_\varepsilon. \tag{3}
\]

Assume that, as \( \varepsilon \to 0^+ \), we have the convergence \( u_\varepsilon \to u \) uniformly on an open set \( \Omega \subseteq \mathbb{R}^n \). Then \( u \) is a viscosity solution of

\[
F(x, u(x), \nabla u(x)) = 0 \quad (HJ)
\]

Proof. Fix \( x \in \Omega \) and assume \( p \in D^- u(x) \).

We need to show that \( F(x, u(x), p) \geq 0 \).

1. By Lemma 1 and Remark 1, there exists \( \varphi \in \mathcal{C}^1 \) with \( \nabla \varphi(x) = p \), \( \varphi(x) = u(x) \) and \( \varphi(y) < u(y) \) for all \( y \neq x \). For any \( \delta > 0 \) we can then find \( 0 < \rho \leq \delta \) and a function \( \psi \in \mathcal{C}^2 \) such that

\[
|\nabla \varphi(y) - \nabla \varphi(x)| \leq \delta \quad \text{if} \quad |y - x| \leq \rho
\]

\[
||\psi - \varphi||_{C^1} \leq \delta
\]

and such that each function \( u_\varepsilon - \psi \) has a local minimum inside the ball \( B(x; \rho) \).
2. Let \( u_\varepsilon - \psi \) have a local minimum at \( x_\varepsilon \in B(x, \rho) \). Then

\[
\nabla \psi(x_\varepsilon) = \nabla u(x_\varepsilon), \quad \Delta u(x_\varepsilon) \geq \Delta \psi(x_\varepsilon).
\]

\[
F(x, u_\varepsilon(x_\varepsilon), \nabla \psi(x_\varepsilon)) \geq \varepsilon \Delta \psi(x_\varepsilon) \tag{4}
\]

3. Extract a convergent subsequence \( x_\varepsilon \to \tilde{x} \in B(x, \rho) \).

Letting \( \varepsilon \to 0^+ \) in (4) we have

\[
F(x, u(\tilde{x}), \nabla \psi(\tilde{x})) \geq 0. \tag{5}
\]

Choosing \( \delta > 0 \) small we can make

\[
|u(\tilde{x}) - u(x)| \quad |\nabla \psi(\tilde{x}) - p|
\]

as small as we like.

By continuity, (5) yields \( F(x, u(x), p) \geq 0 \).

Hence \( u \) is a supersolution. The other half of the proof is similar.
**Comparison Theorems**

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $u_1, u_2 \in C(\overline{\Omega})$ are, respectively, viscosity sub- and supersolutions of

$$u + H(x, Du) = 0 \quad x \in \Omega,$$

and

$$u_1 \leq u_2 \quad \text{on} \quad \partial \Omega. \tag{6}$$

Moreover, assume that the function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$|H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)), \tag{7}$$

where $\omega : [0, \infty[ \to [0, \infty[ \text{ is continuous and non-decreasing, with } \omega(0) = 0$. Then

$$u_1 \leq u_2 \quad \text{on} \quad \overline{\Omega}. \tag{8}$$

**Proof.** Easy case: $u_1, u_2$ smooth.

If (8) fails, then $u_1 - u_2$ attains a positive maximum at $x_0 \in \Omega$. Therefore $p = \nabla u_1(x_0) = \nabla u_2(x_0)$.

By definition of sub- and supersolution:

$$u_1(x_0) + H(x_0, p) \leq 0,$$

$$u_2(x_0) + H(x_0, p) \geq 0. \tag{9}$$

Hence $u_1(x_0) - u_2(x_0) \leq 0$ reaching a contradiction.
In the non-smooth case, we can reach again a contradiction provided that we can find a point $x_0$ such that

(i) $u_1(x_0) > u_2(x_0)$,

(ii) some vector $p$ lies at the same time in the upper differential $D^+ u_1(x_0)$ and in the lower differential $D^- u_2(x_0)$.

A natural candidate for $x_0$ is a point where $u_1 - u_2$ attains a global maximum. However, one of the sets $D^+ u_1(x_0)$ or $D^- u_2(x_0)$ may be empty!

To proceed further, the key observation is that we don’t need to compare values of $u_1$ and $u_2$ at exactly the same point. Indeed, to reach a contradiction, it suffices to find nearby points $x_\varepsilon$ and $y_\varepsilon$ such that

(i') $u_1(x_\varepsilon) > u_2(y_\varepsilon)$,

(ii') some vector $p$ lies at the same time in the upper differential $D^+ u_1(x_\varepsilon)$ and in the lower differential $D^- u_2(y_\varepsilon)$.\[u_1\]

\[u_2\]

\[\Omega\]

\[y_\varepsilon\]

\[x_\varepsilon\]
To find suitable points $x_\varepsilon, y_\varepsilon$:

Look at the function of two variables

$$
\Phi_\varepsilon(x, y) = u_1(x) - u_2(y) - \frac{|x - y|^2}{2\varepsilon}
$$

(10)

This clearly admits a global maximum over the compact set $\overline{\Omega} \times \overline{\Omega}$.

If $u_1 > u_2$ at some point $x_0$, this maximum will be strictly positive.

Taking $\varepsilon > 0$ sufficiently small, the boundary conditions imply that the maximum is attained at some interior point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$.

The points $x_\varepsilon, y_\varepsilon$ must be close to each other, otherwise the penalization term in (10) will be very large and negative.

The function of one single variable

$$
 x \mapsto u_1(x) - \left( u_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} \right) = u_1(x) - \varphi_1(x)
$$

(11)

attains its maximum at the point $x_\varepsilon$. Hence by Lemma 1

$$
\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = \nabla \varphi_1(x_\varepsilon) \in D^+ u_1(x_\varepsilon).
$$

The function of one single variable

$$
 y \mapsto u_2(y) - \left( u_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon} \right) = u_2(y) - \varphi_2(y)
$$

(12)

attains its minimum at the point $y_\varepsilon$. Hence

$$
\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = \nabla \varphi_2(y_\varepsilon) \in D^- u_2(y_\varepsilon).
$$

We have thus discovered two points $x_\varepsilon, y_\varepsilon$ and a vector $p = (x_\varepsilon - y_\varepsilon)/\varepsilon$ which satisfy the conditions (i’)-(ii’).
Proof of Theorem 1.

1. If the conclusion fails, then there exists \( x_0 \in \Omega \) such that
\[
u_1(x_0) - u_2(x_0) = \max_{x \in \Omega} \{u_1(x) - u_2(x)\} \doteq \delta > 0. \tag{11}\]

For \( \varepsilon > 0 \), call \((x_\varepsilon, y_\varepsilon)\) a point where the function
\[
\Phi_\varepsilon(x, y) \doteq u_1(x) - u_2(y) - \frac{|x - y|^2}{2\varepsilon} \tag{10}\]
attains its global maximum on the compact set \(\overline{\Omega} \times \overline{\Omega}\). Clearly,
\[
\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \delta > 0. \tag{12}\]

2. Call \( M \) an upper bound for all values \(|u_1(x)|, |u_2(x)|\), as \( x \in \overline{\Omega} \). Then
\[
\Phi_\varepsilon(x, y) \leq 2M - \frac{|x - y|^2}{2\varepsilon},
\]
\[
\Phi_\varepsilon(x, y) \leq 0 \quad \text{if} \quad |x - y|^2 \geq M\varepsilon.
\]
Hence
\[
|x_\varepsilon - y_\varepsilon| \leq \sqrt{M\varepsilon}. \tag{13}\]

3. Since \( u_1 \leq u_2 \) on the boundary \( \partial \Omega \), for \( \varepsilon > 0 \) small, the maximum in (10) must be attained at some interior point \((x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega\).

4. Consider the smooth functions of one single variable
\[
\varphi_1(x) \doteq u_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon}, \quad \varphi_2(y) \doteq u_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}.
\]

Since \( x_\varepsilon \) provides a local maximum for \( u_1 - \varphi_1 \) and \( y_\varepsilon \) provides a local minimum for \( u_2 - \varphi_2 \), we have
\[
p \doteq \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \in D^+ u_1(x_\varepsilon) \cap D^- u_2(y_\varepsilon). \tag{14}\]

From the definition of viscosity sub- and supersolution we now obtain
\[
u_1(x_\varepsilon) + H(x_\varepsilon, p) \leq 0,
\]
\[
u_2(y_\varepsilon) + H(y_\varepsilon, p) \geq 0. \tag{15}\]
5. Observing that

$$\delta \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leq u_1(x_\varepsilon) - u_2(x_\varepsilon) + \left| u_2(x_\varepsilon) - u_2(y_\varepsilon) \right| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon},$$

we see that

$$\left| u_2(x_\varepsilon) - u_2(y_\varepsilon) \right| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \geq 0$$

and hence, by the uniform continuity of $u_2$,

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0. \quad (16)$$

6. Subtracting the second from the first inequality in (15) we obtain

$$\begin{align*}
\delta & \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \\
& \leq u_1(x_\varepsilon) - u_2(y_\varepsilon) \\
& \leq |H(x_\varepsilon, p) - H(y_\varepsilon, p)| \\
& \leq \omega \left( |x_\varepsilon - y_\varepsilon| \cdot (1 + |x_\varepsilon - y_\varepsilon|^{-1}) \right). \quad (17)
\end{align*}$$

This yields a contradiction, Indeed, by (13) and (16) the right hand side of (17) can be made arbitrarily small by letting $\varepsilon \to 0$.

**Corollary 1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the Hamiltonian function $H$ satisfy the equicontinuity assumption (7). Then the boundary value problem

$$u + H(x, Du) = 0 \quad x \in \Omega$$

$$u = \psi \quad x \in \partial \Omega$$

admits at most one viscosity solution.
Uniqueness for the Cauchy Problem

\[ u_t + H(t, x, Du) = 0 \quad (t, x) \in ]0, T[ \times \mathbb{R}^n, \] (18)
\[ u(0, x) = g(x) \quad x \in \mathbb{R}^n. \] (19)

(H1) $H$ is uniformly continuous on $[0, T] \times \mathbb{R}^n \times K$, for every compact set $K \subset \mathbb{R}^n$.

(H2) There exists a continuous, non-decreasing function $\omega : [0, \infty[ \times [0, \infty[ \to \mathbb{R}$ with $\omega(0) = 0$ such that

\[ |H(t, x, p) - H(t, y, p)| \leq \omega(|x - y|(1 + |p|)). \]

Theorem 2. Let the function $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ satisfy the equicontinuity assumptions (H1)-(H2). Let $u_1, u_2$ be uniformly continuous sub- and super-solutions of (18) respectively. If $u_1(0, x) \leq u_2(0, x)$ for all $x \in \mathbb{R}^n$, then

\[ u_1(t, x) \leq u_2(t, x) \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}^n. \]

Corollary 2. Let the function $H$ satisfy the assumptions (H1)-(H2). Then the Cauchy problem

\[ u_t + H(t, x, Du) = 0 \quad (t, x) \in ]0, T[ \times \mathbb{R}^n \]
\[ u(0, x) = g(x) \quad x \in \mathbb{R}^n \]

admits at most one uniformly continuous viscosity solution $u : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$. 67
Sketch of the proof of Theorem 2.

Assume, on the contrary, that $u_1 > u_2$ at some point $(t, x)$.
Then for some $\lambda > 0$ we still have

$$u_1(t, x) - \lambda t > u_2(t, x)$$

Assume that we can find a global maximum:

$$u_1(t_0, x_0) - \lambda t_0 - u_2(t_0, x_0) = \sup_{t \in [0, T], x \in \mathbb{R}^n} u_1(t, x) - \lambda t - u_2(t, x)$$

Clearly, $t_0 > 0$. If $u_1, u_2$ are smooth, then

$$\nabla u_1(t_0, x_0) = \nabla u_2(t_0, x_0), \quad \partial_t u_1(t_0, x_0) - \lambda \geq \partial_t u_2(t_0, x_0) \tag{20}$$

On the other hand, by assumptions we have

$$\partial_t u_1(t_0, x_0) + H(t_0, x_0, u_1, \nabla u_1) \leq 0$$
$$\partial_t u_2(t_0, x_0) + H(t_0, x_0, u_1, \nabla u_1) \geq 0 \tag{21}$$

Together (20) and (21) yield a contradiction.

Toward the general case - two technical difficulties:

1. The supremum of $u_1 - u_2 - \lambda t$ may only be approached as $|x| \to \infty$.

2. The functions $u_1, u_2$ may not admit upper or lower differentials at the point where the maximum is attained.

Key idea: look at the function of double variables

$$\Phi_\varepsilon(t, x, s, y) = u_1(t, x) - u_2(s, y) - \lambda(t + s) - \varepsilon(|x|^2 + |y|^2) - \frac{1}{\varepsilon}(|x - y|^2 + |t - s|^2)$$

The first penalization term guarantees that a global maximum is attained.

The second penalization term guarantees that, at the point $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ where the maximum is attained, one has $t_\varepsilon \approx s_\varepsilon$ and $x_\varepsilon \approx y_\varepsilon$. 

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Extensions

- Discontinuous solutions
- Second order equations

\[ u : \Omega \mapsto \mathbb{R} \] is upper semicontinuous at \( x_0 \) if
\[
\limsup_{x \to x_0} u(x) \leq u(x_0),
\]

\[ u : \Omega \mapsto \mathbb{R} \] is lower semicontinuous at \( x_0 \) if
\[
\liminf_{x \to x_0} u(x) \geq u(x_0),
\]

A nonlinear operator \( F = F(x, u, Du, D^2 u) \) is (possibly degenerate) elliptic if
\[
F(x, u, p, X) \leq F(x, u, p, X + Y) \quad \text{whenever } Y \geq 0
\]
(i.e. whenever the symmetric matrix \( Y \) is non-negative definite)
General nonlinear elliptic equation:

\[ F(x, u, Du, D^2u) = 0 \]  \hspace{1cm} (22)

An upper semicontinuous function \( u : \Omega \rightarrow \mathbb{R} \) is a **viscosity subsolution** of the degenerate elliptic equation (22) if, for every \( \varphi \in C^2(\Omega) \), at each point \( x_0 \) where \( u - \varphi \) has a local maximum there holds

\[ F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0. \]

A lower semicontinuous function \( u : \Omega \rightarrow \mathbb{R} \) is a **viscosity supersolution** of the degenerate elliptic equation (22) if, for every \( \varphi \in C^2(\Omega) \), at each point \( x_0 \) where \( u - \varphi \) has a local minimum there holds

\[ F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0. \]

Comparison and uniqueness results:


Systems of Hamilton-Jacobi Equations

\[ w_t + H(x, Dw) = 0 \quad w : \Omega \rightarrow \mathbb{R}^n \]

1. Systems of conservation laws in one space dimension

\[ u_t + f(u)_x = 0 \quad (23) \]

\[ u = (u_1, \ldots, u_n) \quad \text{conserved quantities} \]
\[ f = (f_1, \ldots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{fluxes} \]

Since (23) is in divergence form, we can take \( w \) such that

\[ w_x = u \quad w_t = f(u) \]

This yields the H-J system of equations

\[ w_t + f(w_x) = 0 \]
2. Non-cooperative m-persons differential games

\[
\dot{x} = \sum_{i=1}^{m} f_i(x, u_i). \tag{23}
\]

\[ t \mapsto u_i(t) \in U_i \] is the control chosen by the \( i \)-th player, whose aim is to minimize his cost functional

\[
J_i = \int_t^T h_i(x(s), u_i(s)) \, ds + g_i(x(T)). \tag{24}
\]

Let \( V_i(t, x) \) be the expected cost for the \( i \)-th player (if he behaves optimally), if the system starts at state \( x \) at time \( t \).

If such value function \( V = (V_1, \ldots, V_m) \) exists, it should provide a solution to the system of Hamilton-Jacobi equations

\[
\partial_t V_i + H_i(x, \nabla V_1, \cdots, \nabla V_m) = 0, \tag{25}
\]

with terminal data

\[
V_i(T, x) = g_i(x). \tag{26}
\]

The Hamiltonian functions \( H_i \) are defined as follows.

Assume that, for every given \( x \in \mathbb{R}^n \) and \( p_j = \nabla V_j \in \mathbb{R}^n \), there exist an optimal choice \( u_j^*(x, p_j) \in U_j \) for the \( j \)-th player, such that

\[
f_j(x, u_j^*(x, p_j)) \cdot p_j + h_j(x, u_j^*(x, p_j)) = \min_{\omega} \{ f_j(x, \omega) \cdot p_j + h_j(x, \omega) \}
\]

Then the Hamiltonian functions in (25) are

\[
H_i(x, p_1, \ldots, p_m) = \sum_{j=1}^{m} f_j(x, u_j^*(x, p_j)) \cdot p_i + h_i(x, u_i^*(x, p_i))
\]

(No general existence, uniqueness theory is yet available for these systems)
Sufficient Conditions for Optimality

\[ \dot{x} = f(x, u) \quad t \in [0, T], \quad x(0) = x_0 \]  

(1)

Family of admissible control functions: \( U = \{ u : [0, T] \mapsto U \} \)

For \( u(\cdot) \in U \), the trajectory of (1) is \( t \mapsto x(t, u) \)

Terminal constraint

\[ x(T) \in S \] 

(2)

Mayer problem, free terminal point:

\[ \max_{u \in U} \psi(x(T, u)) \] 

(3)

Sufficient Condition: Existence + PMP.

Assume that the optimization problem (1)–(3) admits an optimal solution. Let \( \{u_1, u_2, \ldots, u_N\} \) be the set of all control functions that satisfy the Pontryagin Maximum Principle. If

\[ \psi(x(T, u_k)) = \min_{1 \leq j \leq N} \psi(x(T, u_j)) \]

Then the control \( u_k \) is optimal.
Example 1.

\[(\dot{x}_1, \dot{x}_2) = (u, x_2^2) \quad u \in [-1, 1], \quad (x_1, x_2)(0) = (0, 0)\]

\[
\max_{u \in U} x_2(T) \quad \text{with constraint} \quad x_1(T) = 0
\]

If \(t \mapsto x^*(t) = x(t, u^*)\) is an optimal trajectory, the PMP yields

\[(\dot{p}_1, \dot{p}_2) = (-2x_1p_2, 0), \quad p_2(T) \geq 0\]

\[
p_1u^* + p_2x_1^2 = \max_{\omega \in [-1,1]} p_1 \omega + p_2x_1^2 \quad \implies \quad u^* = \text{sign } p_1
\]

\[
p_2(t) \equiv \bar{p}_2 \geq 0
\]

\[p_2 \equiv 0 \quad \implies \quad p_1 \equiv \bar{p}_1 \quad u^*(t) \equiv 0, \quad x^*(t) \equiv (0, 0)\]
\( p_2 \equiv 1 \quad \implies \quad \dot{p}_1 = -2x_1 \)

\( \ddot{p}_1 = -2 \text{sign} \ p_1, \quad \dot{p}_1(0) = 0 \)

\( \dot{x}_1 = \text{sign} \ p_1 \quad \dot{x}_2 = x_1^2 \)

Countably many controls \( u_j \) satisfy the PMP.

Optimal controls:

\[
  u^*(t) = \begin{cases} 
    1 & \text{if } 0 < t < T/2 \\
    -1 & \text{if } T/2 < t < T 
  \end{cases}
\]

\[
  u^*(t) = \begin{cases} 
    -1 & \text{if } 0 < t < T/2 \\
    1 & \text{if } T/2 < t < T 
  \end{cases}
\]
The Value Function

\[ \dot{x}(t) = f(x(t), \alpha(t)) \quad s < t < T, \]
\[ x(s) = y. \]  \hspace{1cm} (4)\hspace{1cm} (5)

Here \( x \in \mathbb{R}^n \), while the control function \( \alpha : [s, T] \mapsto A \) is required to take values inside a compact set \( A \subset \mathbb{R}^m \).

**Optimization Problem:** select a control \( T \mapsto \alpha^*(t) \) which minimizes the cost
\[
J(s, y, \alpha) \doteq \int_{s}^{T} h(x(t), \alpha(t)) \, dt + g(x(T)).
\]  \hspace{1cm} (6)

**Assumption:** The functions \( f, g, h \) are Lipschitz continuous and uniformly bounded.

Dynamic Programming Approach: study the value function
\[
u(s, y) \doteq \inf_{\alpha(\cdot)} J(s, y, \alpha).
\]  \hspace{1cm} (7)
Bellman’s Dynamic Programming Principle

**Theorem.** For every $s < \tau < T$, $y \in \mathbb{R}^n$, one has

$$u(s, y) = \inf_{\alpha(\cdot)} \left\{ \int_s^\tau h(x(t; s, y, \alpha), \alpha(t)) \, dt + u(\tau, x(\tau; s, y, \alpha)) \right\}. \quad (8)$$

The optimization problem on $[s, T]$ can be split into two separate problems:

- As a first step, we solve the optimization problem on the sub-interval $[\tau, T]$, with running cost $h$ and terminal cost $g$. In this way, we determine the value function $u(\tau, \cdot)$, at time $\tau$.

- As a second step, we solve the optimization problem on the sub-interval $[s, \tau]$, with running cost $h$ and terminal cost $u(\tau, \cdot)$, determined by the first step.

At the initial time $s$, by (8) the value function $u(s, \cdot)$ obtained in step 2 is the same as the value function corresponding to the global optimization problem over the whole interval $[s, T]$.

**Mayer problem:** $h \equiv 0$ (no running cost). In this case one has

- For every admissible trajectory $t \mapsto x(t)$, the value function $t \mapsto u(t, x(t))$ is non-decreasing.

- A trajectory $t \mapsto x^*(t)$ is optimal if and only if $t \mapsto u(t, x^*(t))$ is constant.
**Proof.** Call $J_\tau$ the right hand side of (8).

1. To prove that $J_\tau \leq u(s,y)$, fix $\varepsilon > 0$ and choose a control $\alpha : [s,T] \mapsto A$ such that
   
   \[ J(s,y,\alpha) \leq u(s,y) + \varepsilon. \]

   Observing that
   
   \[ u(\tau, x(\tau; s,y,\alpha)) \leq \int_{\tau}^{T} h(t, x(t; s,y,\alpha)) \, dt + g(x(T; s,y,\alpha)), \]

   we conclude
   
   \[ J_\tau \leq \int_{s}^{\tau} h(t, x(t; s,y,\alpha)) \, dt + u(\tau, x(\tau; s,y,\alpha)) \]
   
   \[ \leq J(s,y,\alpha) \]
   
   \[ \leq u(s,y) + \varepsilon. \]

   Since $\varepsilon > 0$ is arbitrary, this first inequality is proved.

2. To prove that $u(s,y) \leq J_\tau$, fix $\varepsilon > 0$. Then there exists a control $\alpha' : [s,\tau] \mapsto A$ such that
   
   \[ \int_{s}^{\tau} h(t, x(t; s,y,\alpha')) \, dt + u(\tau, x(\tau; s,y,\alpha')) \leq J_\tau + \varepsilon \]

   Moreover, there exists a control $\alpha'' : [\tau,T] \mapsto A$ such that
   
   \[ J(\tau, x(\tau; s,y,\alpha'), \alpha'') \leq u(\tau, x(\tau; s,y,\alpha')) + \varepsilon. \]

   One can now define a new control $\alpha : [s,T] \mapsto A$ as the concatenation of $\alpha', \alpha''$:
   
   \[ \alpha(t) := \begin{cases} 
   \alpha'(t) & \text{if } t \in [s,\tau], \\
   \alpha''(t) & \text{if } t \in [\tau,T]. 
   \end{cases} \]

   This yields
   
   \[ u(s,y) \leq J(s,y,\alpha) \leq J_\tau + 2\varepsilon. \]

   with $\varepsilon > 0$ arbitrarily small.
The Hamilton-Jacobi-Bellman Equation

\[ \dot{x}(t) = f(x(t), \alpha(t)) \quad s < t < T, \]  
\[ x(s) = y, \quad \alpha : [s, T] \mapsto A \] 
\[ J(s, y, \alpha) \doteq \int_s^T h(x(t), \alpha(t)) \, dt + g(x(T)). \] 
\[ u(s, y) \doteq \inf_{\alpha(\cdot)} J(s, y, \alpha). \]

Lemma (Regularity of the Value Function). Let \( f, g, h \) be uniformly bounded and Lipschitz continuous. Then the value function \( u \) is uniformly bounded and Lipschitz continuous on \([0, T] \times \mathbb{R}^n\).

Theorem (Characterization of the Value Function). The value function \( u = u(s, y) \) is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

\[ -\left[ u_t + H(x, \nabla u) \right] = 0 \quad (t, x) \in ]0, T[ \times \mathbb{R}^n, \]

with terminal condition

\[ u(T, x) = g(x) \quad x \in \mathbb{R}^n, \]

and Hamiltonian function

\[ H(x, p) \doteq \min_{a \in A} \{ f(x, a) \cdot p + h(x, a) \}. \]
Proof. To show that \( u \) is a viscosity subsolution of

\[
-\left[ u_t + \min_{a \in A} \left\{ f(x, a) \cdot \nabla u + h(x, a) \right\} \right] = 0
\]  

let \( \varphi \in C^1([0, T] \times \mathbb{R}^n) \). We need to show:

(P1) If \( u - \varphi \) attains a local maximum at a point \((t_0, x_0) \in ]0, T[ \times \mathbb{R}^n\), then

\[
\varphi_t(t_0, x_0) + \min_{a \in A} \left\{ f(x_0, a) \cdot \nabla \varphi(t_0, x_0) + h(x_0, a) \right\} \geq 0. 
\]  

We can assume that

\[
u(t_0, x_0) = \varphi(t_0, x_0), \quad u(t, x) \leq \varphi(t, x) \quad \text{for all } t, x.
\]

If (12) does not hold, then there exists \( a \in A \) and \( \theta > 0 \) such that

\[
\varphi_t(t_0, x_0) + f(x_0, a) \cdot D \varphi(t_0, x_0) + h(x_0, a) < -\theta. 
\]  

We shall derive a contradiction by showing that this control value \( a \) is “too good to be true”. Namely, by choosing a control function \( \alpha(\cdot) \) with \( \alpha(t) \equiv a \) for \( t \in [t_0, t_0 + \delta] \) and such that \( \alpha \) is nearly optimal on the remaining interval \([t_0 + \delta, T]\), we obtain a total cost \( J(t_0, x_0, \alpha) \) strictly smaller than \( u(t_0, x_0) \).
By continuity,
\[ \varphi_t(t_0, x_0) + f(x_0, a) \cdot D\varphi(t_0, x_0) + h(x_0, a) < -\theta \]  
implies
\[ \varphi_t(t, x) + f(x, a) \cdot D\varphi(t, x) < -h(x, a) - \theta \]
whenever \( t \approx t_0 \) and \( x \approx x_0 \). Let \( x(t) = x(t; t_0, x_0, a) \) be the solution of
\[ \dot{x}(t) = f(x(t), a) \quad x(t_0) = x_0. \]
Then
\[
\begin{align*}
  u(t_0 + \delta, x(t_0 + \delta)) - u(t_0, x_0) &\leq \varphi(t_0 + \delta, x(t_0 + \delta)) - \varphi(t_0, x_0) \\
  &= \int_{t_0}^{t_0+\delta} \frac{d}{dt} \varphi(t, x(t)) \, dt \\
  &= \int_{t_0}^{t_0+\delta} \left\{ \varphi_t(t, x(t)) + f(x(t), a) \cdot D\varphi(t, x(t)) \right\} \, dt \\
  &\leq -\int_{t_0}^{t_0+\delta} h(x(t), a) \, dt - \delta\theta 
\end{align*}
\]

On the other hand, the Dynamic Programming Principle
\[
u(t_0, x_0) = \inf_{\alpha(\cdot)} \left\{ \int_{t_0}^{t_0+\delta} h(x(t; t_0, x_0, \alpha), \alpha(t)) \, dt + u(t_0 + \delta, x(t_0 + \delta; t_0, x_0, \alpha)) \right\}
\]
yields
\[
u(t_0 + \delta, x(t_0 + \delta)) - u(t_0, x_0) \geq -\int_{t_0}^{t_0+\delta} h(x(t), a) \, dt.
\]
Together, (15) and (16) yield a contradiction, hence (P1) must hold.
To show that $u$ is a viscosity supersolution of

$$-\left[u_t + \min_{a \in A} \{f(x,a) \cdot \nabla u + h(x,a)\}\right] = 0$$  \hspace{1cm} (9)

let $\varphi \in C^1([0,T[ \times \mathbb{R}^n)$.

We need to show:

(P2) If $u - \varphi$ attains a local minimum at a point $(t_0, x_0) \in ]0, T[ \times \mathbb{R}^n$, then

$$\varphi_t(t_0, x_0) + \min_{a \in A} \{f(x_0,a) \cdot \nabla \varphi(t_0, x_0) + h(x_0,a)\} \leq 0.$$  \hspace{1cm} (17)

We can assume that

$$u(t_0, x_0) = \varphi(t_0, x_0), \quad u(t, x) \geq \varphi(t, x) \quad \text{for all } t, x.$$  \hspace{1cm} (18)

If (P2) fails, then there exists $\theta > 0$ such that

$$\varphi_t(t_0, x_0) + f(x_0,a) \cdot D\varphi(t_0, x_0) + h(x_0,a) > \theta \quad \text{for all } a \in A.$$  \hspace{1cm} (19)

In this case, we shall reach a contradiction by showing that no control function $\alpha(\cdot)$ is good enough. Namely, whatever control function $\alpha(\cdot)$ we choose on the initial interval $[t_0, t_0 + \delta]$, even if during the remaining time $[t_0 + \delta, T]$ our control is optimal, the total cost will still be considerably larger than $u(t_0, x_0)$.
By continuity,

\[ \varphi_t(t_0, x_0) + f(x_0, a) \cdot D \varphi(t_0, x_0) + h(x_0, a) > \theta \quad \text{for all } a \in A \quad (17) \]

implies

\[ \varphi_t(t, x) + f(x, a) \cdot D \varphi(t, x) > \theta - h(x, a) \quad \text{for all } a \in A \quad (20) \]

for all \( t \approx t_0 \) and \( x \approx x_0 \). Choose an arbitrary control function \( \alpha : [t_0, t_0+\delta] \mapsto A \), and call \( t \mapsto x(t) = x(t; t_0, x_0, \alpha) \) the corresponding trajectory. We now have

\[
\begin{align*}
&u(t_0 + \delta, x(t_0 + \delta)) - u(t_0, x_0) \geq \varphi(t_0 + \delta, x(t_0 + \delta)) - \varphi(t_0, x_0) \\
&\quad = \int_{t_0}^{t_0+\delta} \frac{d}{dt} \varphi(t, x(t)) \, dt \\
&\quad = \int_{t_0}^{t_0+\delta} \varphi_t(t, x(t)) + f(x(t), \alpha(t)) \cdot D \varphi(t, x(t)) \, dt \\
&\quad \geq \int_{t_0}^{t_0+\delta} \theta - h(x(t), \alpha(t)) \, dt
\end{align*}
\]

(21)

Therefore, for every control function \( \alpha(\cdot) \),

\[
\begin{align*}
&u(t_0 + \delta, x(t_0 + \delta)) + \int_{t_0}^{t_0+\delta} h(x(t), \alpha(t)) \, dt \geq u(t_0, x_0) + \delta \theta. \quad (22)
\end{align*}
\]

Taking the infimum of the left hand side of (22) over all control functions \( \alpha \), we see that this infimum is still \( \geq u(t_0, x_0) + \delta \theta \).

On the other hand, the Dynamic Programming Principle states that

\[
u(t_0, x_0) = \inf_{\alpha(\cdot)} \left\{ \int_{t_0}^{t_0+\delta} h(x(t; t_0, x_0, \alpha), \alpha(t)) \, dt + u(t_0 + \delta, x(t_0 + \delta; t_0, x_0, \alpha)) \right\}
\]

in contradiction with (22). This establishes (P2).

Therefore, \( u \) is a \textbf{viscosity solution}. 

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Sufficient Conditions for Optimality

1. For each initial condition \((s, y)\), construct a “candidate” optimal control \(\alpha^{s,y} : [s, T] \mapsto A\), satisfying the PMP.

2. Consider the corresponding cost

\[
u(s, y) = J(s, y, \alpha^{s,y}) \quad (23)\]

3. Check whether \(v\) satisfies the Hamilton-Jacobi-Bellman equation

\[
-\left[ u_t + \min_{a \in A} \left\{ f(x, a) \cdot \nabla u + h(x, a) \right\} \right] = 0 \quad (t, x) \in ]0, T[ \times \mathbb{R}^n, \quad (9)
\]

in the viscosity sense, with terminal condition

\[
u(T, x) = g(x) \quad x \in \mathbb{R}^n \quad (10)
\]

In the positive case, by uniqueness, \(u\) coincides with the minimum value function. Hence all controls \(\alpha^{s,y}\) are optimal.

The optimal control is found in feedback form:

\[
\alpha^*(t, x) = \arg\min_{a \in A} \left\{ f(x, a) \cdot \nabla u + h(x, a) \right\}
\]

at every point \((t, x)\) where \(u\) is differentiable.
Regular Synthesis

For the optimization problem (4)–(6), let \( u = u(t,x) \) be the minimum value function, defined at (7).

**Theorem.** Let \( v : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R} \) be a continuous function such that

\[
v \geq u
\]

\[
v(T,x) = g(x) \quad x \in \mathbb{R}^n.
\]

Assume that there exist \( C^1 \) manifolds \( \mathcal{M}_1, \ldots, \mathcal{M}_N \subset [0,T] \times \mathbb{R}^n \) of dimension \( \leq n \), such that \( v \) is continuously differentiable and satisfies

\[
v_t + \min_{a \in A} \{ f(x,a) \cdot \nabla v + h(x,a) \} = 0
\]

on the open set \( ]0,T[ \times \mathbb{R}^n \setminus \bigcup \mathcal{M}_j \).

Then \( v = u \) on \( [0,T] \times \mathbb{R}^n \).

**Remark.** Here we only require (9) to be satisfied on an open set where \( v \) is \( C^1 \). Nothing has to be checked at points \((t,x) \in \mathcal{M}_j\).

On the other hand, we assume that \( v \) is sufficiently regular, i.e. \( v \in C^1 \) outside finitely many manifolds.

In this case, we say that the optimization problem admits a **regular synthesis**.

**Proof.** We only need to show that \( u \geq v \).

1. Assume \( u(t_0,x_0) < v(t_0,x_0) \). By continuity,

\[
u(t,x) + 3\varepsilon \leq v(t,x)
\]

for some \( \varepsilon > 0 \) and all \((t,x)\) in a neighborhood of \((t_0,x_0)\).
2. Since $u$ is the infimum of the cost, there exists a control $\alpha^* : [t_0, T] \mapsto A$ such that the trajectory $t \mapsto x^*(t)$ satisfies

$$x^*(t_0) = x_0, \quad \dot{x}^*(t) = f(x^*(t), \alpha^*(t)) \quad \text{for a.e. } t \in [t_0, T]$$

$$\int_{t_0}^{T} h(x^*(t), \alpha^*(t)) \, dt + g(x^*(T)) \leq u(t_0, x_0) + \varepsilon \quad (25)$$

3. The set of piecewise constant controls is $L^1$ dense on the set of all measurable controls $\alpha : [t_0, T] \mapsto A$. Hence we can find a piecewise constant control $\alpha^\clubsuit$ such that, calling $t \mapsto x^\clubsuit(t)$ the solution to

$$\dot{x}^\clubsuit(t) = f(x^\clubsuit(t), \alpha^\clubsuit(t)), \quad x^\clubsuit(T) = x^*(T),$$

also $\alpha^\clubsuit$ is nearly optimal. Its cost is

$$J(\alpha^\clubsuit) = \int_{t_0}^{T} h(x^\clubsuit(t), \alpha^\clubsuit(t)) \, dt + g(x^\clubsuit(T)) \leq u(t_0, x^\clubsuit(t_0)) + 2\varepsilon \quad (26)$$

We shall obtain a contradiction by showing that the total cost for $\alpha^\clubsuit$ is

$$J(\alpha^\clubsuit) \geq v(t_0, x^\clubsuit(t_0)) \geq u(t_0, x^\clubsuit(t_0)) + 3\varepsilon$$
4. Let \( t_0 < t_1 < \cdots < t_m = T \) be the times where \( u^\bullet \) has a jump.

If we show that, for every \( j = 1, \ldots, m \)
\[
\int_{t_{j-1}}^{t_j} h(x^\bullet(t), u^\bullet(t)) \, dt \geq v(t_{j-1}, x^\bullet(t_{j-1})) - v(t_j, x^\bullet(t_j)) \tag{27}
\]
we reach a contradiction. Indeed, summing over \( j \) we obtain
\[
\int_{t_0}^{T} h(x^\bullet(t), u^\bullet(t)) \, dt \geq v(t_0, x^\bullet(t_0)) - v(T, x^\bullet(T)).
\]
Then the total cost using the control \( \alpha^\bullet \) is
\[
\int_{t_0}^{T} h(x^\bullet(t), u^\bullet(t)) \, dt + g(x^\bullet(T)) \geq v(t_0, x^\bullet(t_0)). \tag{28}
\]

5. It remains to prove (27). More generally we show that, for every constant control \( \alpha(t) \equiv \tilde{a} \) and every trajectory
\[
\dot{x}(t) = f(x(t), \tilde{a}) \quad t \in [\tau, \tau']
\]
we have
\[
\int_{\tau}^{\tau'} h(x(t), \tilde{a}) \, dt \geq v(\tau, x(\tau)) - v(\tau', x(\tau')). \tag{29}
\]
By a transversality argument, we can find a sequence of trajectories
\[ \dot{x}_m = f(x_m(t), \tilde{a}) \]
uniformly converging to \( x \), such that each \( x_m \) crosses each manifold \( M_j \) only at finitely many times. Since \( x_m(t) \not\in \bigcup M_j \) outside these crossing times, the Hamilton-Jacobi-Bellman equation
\[ v_t + \min_{a\in A} \left\{ f(x, a) \cdot \nabla v + h(x, a) \right\} = 0 \quad (9) \]
yields
\[ \frac{d}{dt} v(t, x_m(t)) + h(x_m, \tilde{a}) = v_t + \nabla v \cdot f(x_m, \tilde{a}) + h(x_m, \tilde{a}) \geq 0 \]
Therefore, integrating over \([\tau, \tau']\)
\[ \int_{\tau}^{\tau'} h(x_m(t), \tilde{a}) \, dt + v(\tau', x_m(\tau')) - v(\tau, x_m(\tau)) \geq 0 \]
Letting \( m \to \infty \) we conclude
\[ \int_{\tau}^{\tau'} h(x(t), \tilde{a}) \, dt \geq v(\tau, x(\tau)) - v(\tau', x(\tau')). \quad (29) \]
Nonlinear First Order P.D.E.

The Method of Characteristics

\[ v_t + H(x, \nabla v) = 0 \]  \hfill (30)

\[ v(\tau, x) = \bar{v}(x), \]  \hfill (31)

Assume \( H, v \) smooth. Call \( p = \nabla v \), so that \( p = (p_1, \ldots, p_n) = (v_{x_1}, \ldots, v_{x_n}) \)

\[ \frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}. \]

Differentiating (30) w.r.t. \( x_i \) one obtains

\[ \frac{\partial p_i}{\partial t} = \frac{\partial^2 v}{\partial x_i \partial t} = -\frac{\partial H}{\partial x_i} - \sum_j \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial x_j}. \]

The total derivative of \( p_i \) along a curve \( t \mapsto x(t) \) is

\[ \frac{d}{dt} p_i(t, x(t)) = \frac{\partial p_i}{\partial t} + \sum_j \dot{x}_j \frac{\partial p_i}{\partial x_j} \]

\[ = -\frac{\partial H}{\partial x_i} + \sum_j \left( \dot{x}_j - \frac{\partial H}{\partial p_j} \right) \frac{\partial p_i}{\partial x_j}. \]  \hfill (32)

By choosing \( \dot{x} = \partial H/\partial p \), the last term in (32) disappears.
Method of characteristics:

For each $\bar{x}$, solve the Hamiltonian system of O.D.E's

$$\begin{cases} 
\dot{x}_i = \frac{\partial H}{\partial p_i}(x, p), \\
\dot{p}_i = -\frac{\partial H}{\partial x_i}(x, p), \\
x_i(\tau) = \bar{x}_i, \\
p_i(\tau) = \frac{\partial \bar{v}}{\partial x_i}(\bar{x}).
\end{cases} \tag{33}$$

Write the solution as $t \mapsto x(t, \bar{x})$, $t \mapsto p(t, \bar{x})$.

Observing that $\nabla v(t, x(t, \bar{x})) = p(t, \bar{x})$ and

$$\frac{d}{dt} v(t, x(t, \bar{x})) = v_t + \nabla v \cdot \dot{x} = -H(x, p) + p \cdot \frac{\partial H}{\partial p},$$

we obtain

$$v(t, x(t, \bar{x})) = v(\tau, \bar{x}) + \int_{\tau}^{t} \left( -H(x, p) + p \cdot \frac{\partial H}{\partial p} \right) ds$$
Pontryagin’s Maximum Principle and the P.D.E. of Dynamic Programming

\[ \dot{x} = f(x, \alpha) \quad \alpha(t) \in A \]  \hspace{1cm} (34)

Minimize a cost functional involving a running cost \( h \) and a terminal cost \( g \):

\[ J^{s,y}(\alpha) = \int_{s}^{T} h(x(t), \alpha(t)) \, dt + g(x(T)). \]  \hspace{1cm} (35)

Here \( t \mapsto x(t) \) is the trajectory of (34) corresponding to the control \( \alpha : [s, T] \mapsto A \) and with initial data \( x(s) = y \).

**Value function:**

\[ v(t, x) = \inf_{\alpha(\cdot)} J^{t,x}(\alpha) \]  \hspace{1cm} (36)

is a viscosity solution of the Hamilton-Jacobi-Bellman equation

\[ -[v_t + H(x, \nabla v)] = 0 \]  \hspace{1cm} (37)

with

\[ H(x, p) = \min_{a \in A} \{ p \cdot f(x, a) + h(x, a) \}. \]  \hspace{1cm} (38)

On a region where \( v \) is smooth, the H-J equation (37) can be solved by the method of characteristics. Assume

\[ H(x, p) = p \cdot f(x, a^*(x, p)) + h(x, a^*(x, p)) = \min_{a \in A} \{ p \cdot f(x, a) + h(x, a) \}. \]  \hspace{1cm} (38)

At the point \( \alpha^* \) where the minimum is attained, one has

\[ p \cdot \frac{\partial f}{\partial a}(x, \alpha^*) + \frac{\partial h}{\partial a}(x, \alpha^*) = 0. \]

Hence the Hamiltonian system (33) takes the form

\[
\begin{cases}
\dot{x} = f(x, \alpha^*(x, p)) \\
\dot{p} = -p \cdot \frac{\partial f}{\partial x}(x, \alpha^*(x, p)) - \frac{\partial h}{\partial x}(x, \alpha^*(x, p))
\end{cases}
\]  \hspace{1cm} (39)

The existence of an adjoint vector \( t \mapsto p(t) \) satisfying the linear evolution equation in (39) and the minimality condition (38) is a well known necessary condition for optimality, stated in the **Pontryagin Maximum Principle**.

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References

• General theory of differential inclusions


• Control systems: basic theory and the Pontryagin Maximum Principle


• Geometric properties of reachable sets, local controllability


• Existence of optimal controls


• Nonlinear Bang-Bang Theorems

• Extensions of the Pontryagin Maximum Principle


• Viscosity solutions to Hamilton-Jacobi equations


• Regular feedback synthesis, sufficient conditions for optimality
