Notes on the Boltzmann Equation

Alberto Bressan - Dept. of Mathematics, Penn State University

(Lecture notes for a summer course given at S.I.S.S.A., Trieste, 2005)

We wish to describe the motion of a rarefied gas, consisting of a very large number of identical particles, moving in a three-dimensional space. For $(t, x, \xi) \in [0, \infty[\times I\!R^3 \times I\!R^3$, we consider a function $f(t, x, \xi)$ describing the **density of particles** at time t, at the point x, having speed ξ. In alternative, we may also think of $f(t, x, \xi)$ as the probability of finding a particle with speed ξ near the point x, at time t.
If no collisions occur, the speed ξ of each particle will remain constant in time. A particle with speed ξ and located at the point x at the initial time $t = 0$ will move to $x + \tau\xi$ at a later time τ. Therefore $f(\tau, x, \xi) = f(0,\ x - \tau\xi,\ \xi)$. In this case, f provides a solution to the linear transport equation

$$\partial_t f + \xi \cdot \nabla_x f = 0. \tag{0.1}$$

The presence of collisions accounts for an additional quadratic term on the right hand side:

$$\partial_t f + \xi \cdot \nabla_x f = Q(f). \tag{0.2}$$

The equation (0.2) is the famous **Boltzmann equation**. A specific form of the collision kernel Q will be derived in the next section.

1 - The collision kernel.

In the following, the particles of the gas will be modelled as hard spheres, all with the same radius, that hit each other with perfectly elastic collisions.

We first observe that if two particles move along the same straight line and hit each other with a perfectly elastic collision, their velocities will be exchanged after the impact (fig. 1).

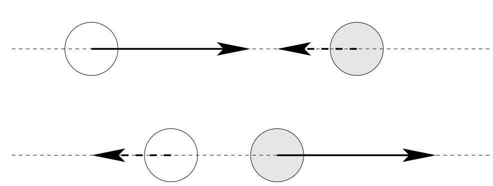

figure 1

More generally, if two spherical particles move in $I\!R^3$, by their **angle of collision** we mean the unit vector $\mathbf{n}$ parallel to the segment joining the centers of the spheres at the instant of collision.

If the particles hit each other at an angle $\mathbf{n}$, then the components of their velocities along $\mathbf{n}$ will be exchanged, while the components perpendicular to $\mathbf{n}$ will remain the same (fig. 2). Calling ξ and ξ_* the velocities of the particles before the collision, and ξ', ξ'_* their respective velocities after the collision, we thus have

$$\begin{cases} \xi' = \xi - \big(\mathbf{n}\cdot(\xi-\xi_*)\big)\,\mathbf{n}, \\ \xi'_* = \xi_* + \big(\mathbf{n}\cdot(\xi-\xi_*)\big)\,\mathbf{n}. \end{cases} \tag{1.1}$$

The exchange in the normal velocities is reflected by the identities

$$\xi'\cdot\mathbf{n} = \xi_*\cdot\mathbf{n}\,, \qquad\qquad \xi'_*\cdot\mathbf{n} = \xi\cdot\mathbf{n}\,.$$

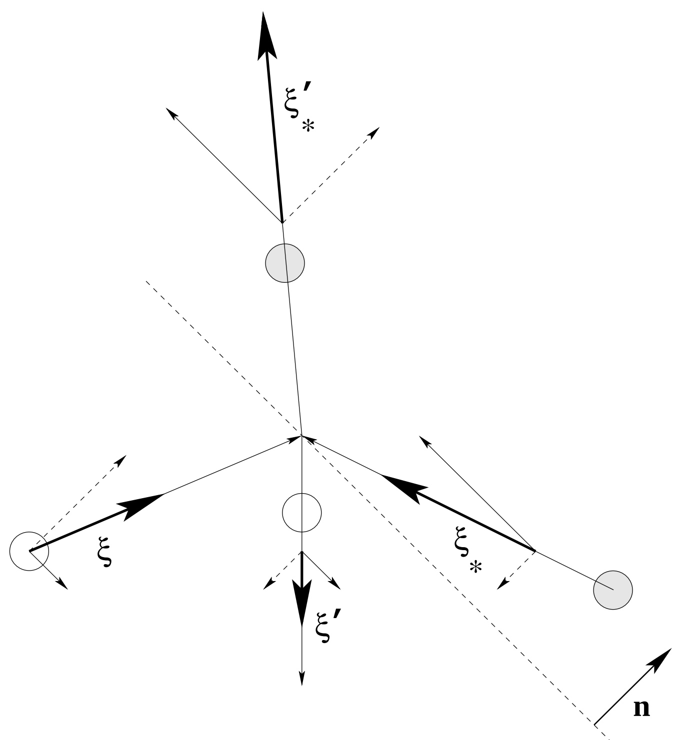

figure 2

Notice that a change in the sign of $\mathbf{n}$ does not affect the linear transformation (1.1). Setting

$$V \doteq \xi - \xi_*\,, \qquad\qquad \xi_0 \doteq \frac{\xi+\xi_*}{2}\,, \qquad\qquad \rho \doteq \frac{|\xi-\xi_*|}{2}\,, \tag{1.2}$$

an alternative construction of the incoming and outgoing speeds is illustrated in fig. 3. Consider the sphere

$$S_{\xi\xi_*} \doteq \big\{\xi' \in I\!R^3\,;\ \ |\xi'-\xi_0| = \rho\big\}$$

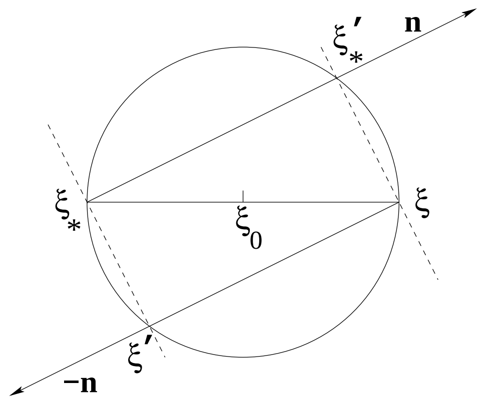

figure 3

i.e. the surface of the ball $B(\xi_0,\, \rho)$ having the segment $\xi\xi_*$ as a diameter. If the collision occurs at an angle $\mathbf{n}$, the speeds ξ', ξ'_* of the two outgoing particles are obtained as the intersections of the sphere $S_{\xi\xi_*}$ with the lines through ξ, ξ_*, parallel to $\mathbf{n}$.

Let now $S^2 \doteq \{\mathbf{n} \in I\!R^3\,;\quad |\mathbf{n}| = 1\}$ be the surface of the unit ball. To find the probability that the collision occurs at an angle $\mathbf{n} \in S^2$, it is convenient to take a frame of reference in which one particle is at rest, say $\xi_* = 0$, while the other has speed $V = \xi - \xi^*$. Instead of thinking of two rigid spheres both with radius $r > 0$, we can equivalently think of a point particle hitting a sphere with radius $2r$. In this case (fig. 4), we see that the rate of collisions at an angle $\mathbf{n}$ is given by $|\mathbf{n} \cdot V|\, d\mathbf{n}$, where $d\mathbf{n}$ denotes the surface element on the unit sphere S^2.

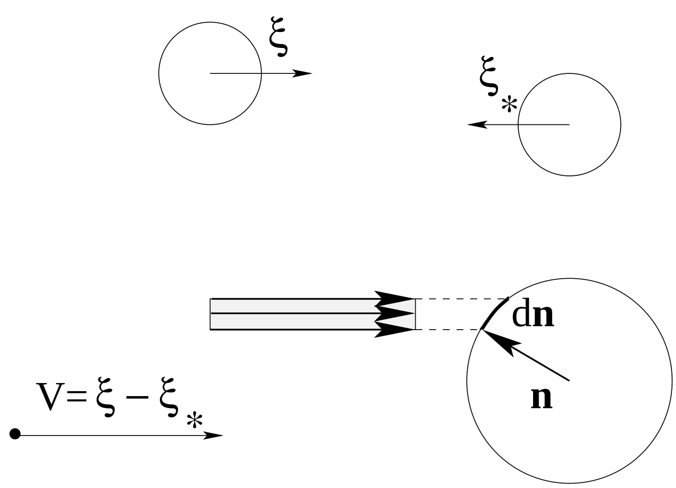

figure 4

Following the probabilistic interpretation, we see that the collision kernel has the effect of replacing the two particles with speeds ξ, ξ_* with a cloud of particles whose speeds are distributed over the sphere $S_{\xi\xi_*}$ having the segment $\xi\xi_*$ as diameter. Actually, an elementary computation shows that the speed of the outgoing particles is *uniformly distributed over the sphere* $S_{\xi\xi_*}$. To see this, referring to fig. 5 we let $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ be the spherical coordinates of the angle at which the collision takes place, and let (θ', φ') be the spherical coordinates of the outgoing velocity ξ' on the sphere $S_{\xi\xi_*}$. Choosing a oordinate frame so that the x-axis is parallel to the vector $\xi_* - \xi$, we thus have

$$\mathbf{n} = (\cos\theta\,,\ \sin\theta\,\cos\varphi\,,\ \sin\theta\,\sin\varphi),$$

$$\xi' - \xi_0 = \rho\,(\cos\theta'\,,\ \sin\theta'\,\cos\varphi'\,,\ \sin\theta'\,\sin\varphi').$$

Notice that (1.1) yields

$$\theta' = 2\theta\,, \qquad\qquad \varphi' = \varphi\,.$$

We observe that the probability that the collision angle $\mathbf{n}$ lies in the infinitesimal region described by $dA = [\theta,\ \theta + d\theta] \times [\varphi,\ \varphi + d\varphi]$ is

$$\text{Prob.}(dA) = \cos\theta\, d\mathbf{n} = \cos\theta\, \sin\theta\, d\theta d\varphi\,,$$

where $d\mathbf{n}$ is the area of the infinitesimal region. Hence, the area covered by the corresponding speed ξ' is

$$\text{meas}(dA') = \rho^2 d\theta' \sin\theta' d\varphi' = \rho^2\, 2\theta d\theta \cdot 2\sin\theta\cos\theta\, d\varphi = 4\rho^2 \text{Prob.}(dA)\,. \tag{1.3}$$

According to (1.3), the area of any portion dA' of the spherical surface $S_{\xi\xi_*}$ is proportional to the probability Prob.$\{\xi' \in dA'\}$. This proves the uniform distribution of the outgoing velocities.

We remark that, as the intensity of collisions grows as $|\xi - \xi_*|$ and the surface of the sphere is $4\pi\rho^2 = \pi\,|\xi - \xi_*|^2$, it is clear that the above probability density per unit area decreases as ρ^{-1}.

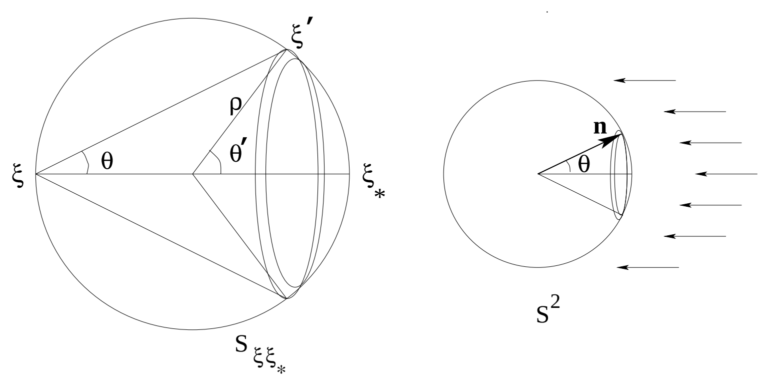

figure 5

We now consider one particular speed $\xi \in I\!R^3$. The rate at which particles with speed ξ will hit other particles and thus change their speed is

$$\begin{aligned} Q_-(f)(\xi) &= \alpha \int_{I\!R^3}\int_{S^2} \big|\mathbf{n}\cdot(\xi - \xi_*)\big|\, f(\xi)f(\xi_*)\, d\mathbf{n}\, d\xi_* \\ &= 2\pi\alpha \int_{I\!R^3} |\xi - \xi_*|\, f(\xi)f(\xi_*)\, d\xi_*\,, \end{aligned} \tag{1.4}$$

where α is some constant factor and $d\mathbf{n}$ denotes the surface element on the unit sphere. This is a *loss term*, responsible for the decrease in the number of particles with speed ξ.

Remark 1. As shown in fig. 4, the integral in (1.4) should actually range only over values of $\mathbf{n}$ in the half sphere

$$S_- \doteq \{\mathbf{n} \in S^2\,; \quad \mathbf{n}\cdot(\xi - \xi_*) < 0\}\,.$$

However, extending the integral over the whole sphere S we obtain exactly the same result, multiplied by two. It is thus convenient to integrate over S, and incorporate the factor $1/2$ within the constant of proportionality α.

Next, we observe that new particles with speed ξ may emerge from interactions. Indeed, since the collision dynamics is perfectly reversible, a new particle with speed ξ will be produced by the interaction of two particles with speeds ξ', ξ'_* as in (1.1), at an angle $\mathbf{n}$. Indeed, such collision would exchange once again the components along $\mathbf{n}$ of the two speeds, thus yielding the original two particles with speeds ξ, ξ_*. Notice that, for any given $\xi_* \in I\!R^3$ and any fixed angle $\mathbf{n} \in S^2$, the equations (1.1) uniquely determine the speeds of two particles whose collision yields the desired result. The rate at which such collisions occur, between particles of speeds ξ', ξ'_*, is proportional to

$$\big|\mathbf{n}\cdot(\xi' - \xi'_*)\big|\, f(\xi')f(\xi'_*) = \big|\mathbf{n}\cdot(\xi - \xi_*)\big|\, f(\xi')f(\xi'_*). \tag{1.5}$$

The above identity is an immediate consequence of (1.1). Integrating over all vectors $\xi_* \in I\!R^3$ and all angles $\mathbf{n} \in S^2$, we find that new particles with speed ξ are created at the rate

$$Q_+(f)(\xi) = \alpha \int_{I\!R^3}\int_{S^2} \big|\mathbf{n}\cdot(\xi - \xi_*)\big|\, f(\xi')f(\xi'_*)\, d\mathbf{n}\, d\xi_*\,. \tag{1.6}$$

This is a *gain term*, responsible for the increase in the density of particles with speed ξ.

By a coordinate rescaling, we can assume that the proportionality constant in (1.4) and (1.6) is $\alpha = 1$. The Boltzmann equation in (0.2) thus takes the form

$$\partial_t f + \xi\cdot\nabla_x f = \int_{I\!R^3}\int_{S^2} (f'f'_* - ff_*)\,\big|\mathbf{n}\cdot(\xi - \xi_*)\big|\, d\mathbf{n}\, d\xi_*\,. \tag{1.7}$$

It is here understood that

$$\begin{aligned} f &= f(t,x,\xi), \qquad & f' &= f(t,x,\xi'),\\ f_* &= f(t,x,\xi_*), \qquad & f'_* &= f(t,x,\xi'_*), \end{aligned} \tag{1.8}$$

with ξ', ξ'_* given at (1.1) in terms of ξ, ξ_* and $\mathbf{n}$.

2 - Collision Invariants.

In this section we consider the homogeneous Boltzmann equation

$$\partial_t\, f = Q(f) \doteq \int_{I\!R^3}\int_{S^2} (f'f'_* - ff_*)\,\big|\mathbf{n}\cdot(\xi - \xi_*)\big|\, d\mathbf{n}\, d\xi_* \tag{2.1}$$

where $f = f(t,\xi)$ is independent of x. We seek functionals of the form

$$\Phi(f) \doteq \int_{I\!R^3} \phi(\xi)\, f(\xi)\, d\xi \tag{2.2}$$

which remain constant in time, along every solution of (2.1). Of course, this holds provided that

$$\int_{\mathbb{R}^3} \phi(\xi)\, Q(f)\, d\xi = 0 \tag{2.3}$$

for every velocity distribution f. Observing that the collision kernel Q replaces two particles with speed ξ, ξ_* with a cloud of particles uniformly distributed on the sphere $S_{\xi\xi_*}$, we see that (2.3) holds for every $f \in C_c^\infty$ if and only if

$$\phi(\xi) + \phi(\xi_*) = \int_{S_{\xi\xi_*}} \phi(\xi')\, d\sigma \tag{2.4}$$

where $d\sigma$ denotes the normalized surface area on the sphere $S_{\xi\xi_*}$. From (2.4) we deduce

$$\phi(\xi) + \phi(\xi_*) = \phi(\xi') + \phi(\xi'_*) \tag{2.5}$$

whenever the segments $\xi\xi_*$ and $\xi'\xi'_*$ are diameters of the same sphere, i.e. whenever (1.1) holds.

Lemma 2.1. *Let $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ be a $\mathcal{C}^2$ function that satisfies (2.5). Then*

$$\phi(\xi) = a + b \cdot \xi + c|\xi|^2 \tag{2.6}$$

for suitable constants $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$.

Proof. Set

$$a \doteq \phi(0)\,, \qquad b \doteq \nabla\phi(0)\,, \qquad c \doteq \frac{1}{6}\,\Delta\phi(0)\,.$$

We need to show that

$$\psi(\xi) = \phi(\xi) - a - b \cdot \xi - c\,|\xi|^2 \equiv 0\,.$$

Notice that ψ is a collision invariant and satisfies

$$\psi(0) = 0\,, \qquad \nabla\psi(0) = 0\,, \qquad \Delta\psi(0) = 0\,.$$

Consider any ξ_* and let $\mathbf{n}$ be any unit vector perpendicular to $\xi\xi_*$ (see fig. 6a). Using (2.5) with $\xi = s\mathbf{n}$, $\xi' = 0$ and $\xi'_* = \xi_* + s\mathbf{n}$, for every s we obtain

$$\psi(s\,\mathbf{n}) + \psi(\xi_*) = \psi(0) + \psi(\xi_* + s\,\mathbf{n})\,.$$

Letting $s \to 0+$ we compute the directional derivative $\nabla\psi(\xi_*)\cdot\mathbf{n} = 0$. This implies that $\psi = \psi\big(|\xi|\big)$ must be a radially symmetric function. Its Taylor expansion at the origin vanishes up to second order.

Next, fix any unit vector $\mathbf{n}$, set $\xi = s\mathbf{n}$, $\xi_* = r\mathbf{n}$, and choose ξ', ξ'_* so that the segment $\xi'\xi'_*$ is perpendicular to $\xi\xi_*$ (see fig. 6b). For $0 < s < r$ we obtain

$$\psi(r) + \psi(s) = 2\psi\big(\rho(r,s)\big)\,, \tag{2.7}$$

where

$$\rho(r,s) = \sqrt{\left(\frac{r-s}{2}\right)^2 + \left(\frac{r+s}{2}\right)^2} = \sqrt{\frac{r^2+s^2}{2}}\,.$$

We now keep r fixed and let s vary. Differentiating (2.7) twice w.r.t. s, at $s = 0$ we find

$$\frac{\partial}{\partial s}\psi\big(\rho(r,s)\big) = \psi'(\rho)\,\frac{1}{2\sqrt{\frac{r^2+s^2}{2}}}\cdot s\,,$$

$$\frac{\partial^2}{\partial s^2}\psi\big(\rho(r,s)\big)\bigg|_{s=0} = \psi'\left(\frac{r}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}\,r} = 0$$

The computation of the radial derivative thus yields $\psi'(r/\sqrt{2}) \equiv 0$, hence $\psi \equiv 0$, as claimed. □

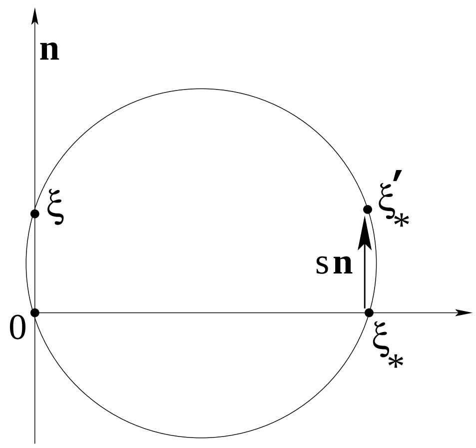

figure 6a

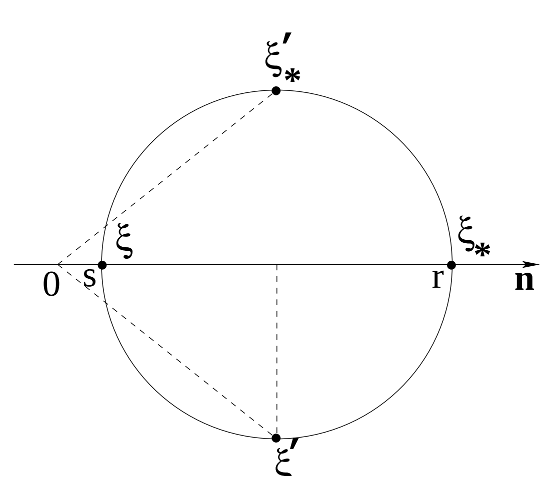

figure 6b

If now ϕ is a collision invariant, then the functional Φ in (2.2) is constant in time. These invariants have a clear physical meaning:

1. Taking $\phi(\xi) \equiv 1$ one obtains the conservation of

$$\int_{\mathbb{R}^3} f(\xi)\, d\xi \qquad \text{(mass)} \tag{2.11}$$

2. Taking $\phi(\xi) \doteq \mathbf{e}_i \cdot \xi, \quad i = 1,2,3$ (with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the standard basis in $\mathbb{R}^3$), we obtain the conservation of the vector

$$\int_{\mathbb{R}^3} \xi\, f(\xi)\, d\xi \qquad \text{(momentum)} \tag{2.12}$$

3. The choice $\phi(\xi) \doteq |\xi|^2/2$ yields the conservation of

$$\int_{\mathbb{R}^3} \frac{|\xi|^2}{2} f(\xi)\, d\xi \qquad \text{(energy)} \tag{2.13}$$

3 - Maxwellian distributions

We investigate here the existence of velocity distributions $f = f(\xi)$ which yield a vanishing collision integral:

$$Q(f)(\xi) \doteq \int_{\mathbb{R}^3} \int_{S^2} (f' f'_* - f f_*) \left|\mathbf{n} \cdot (\xi - \xi_*)\right| d\mathbf{n}\, d\xi_* = 0 \qquad \xi \in \mathbb{R}^3 . \tag{3.1}$$

Toward this goal, using the identity (2.8) with $f = g$ and $\phi(\xi) \doteq \log f(\xi)$, we obtain the **Boltzmann inequality**

$$\begin{aligned}\int_{I\!R^3} Q(f)\,\log f\,d\xi &= \frac{1}{2}\int_{I\!R^3}\int_{I\!R^3}\int_{S^2}(f'f'_* - ff_*)\big(\log f + \log f_* - \log f' - \log f'_*\big)\,|\mathbf{n}\cdot V|\,d\mathbf{n}\,d\xi_*\,d\xi\\ &= \frac{1}{2}\int_{I\!R^3}\int_{I\!R^3}\int_{S^2}(f'f'_* - ff_*)\,\log(ff_*/f'f'_*)\,|\mathbf{n}\cdot V|\,d\mathbf{n}\,d\xi_*\,d\xi\\ &\leq 0\end{aligned} \tag{3.2}$$

because of the elementary inequality

$$(z-y)\,\log(y/z) \leq 0 \qquad\qquad \text{for all } \; y,z>0\,. \tag{3.3}$$

Notice that the equality holds in (3.3) if and only if $y = z$. As a consequence, the left hand side of (3.2) is zero if and only if

$$ff_* = f'f'_*\,. \tag{3.4}$$

Taking the logarithms of both sides, we see that (3.4) holds if and only if $\phi \doteq \log f$ satisfies (2.9). The set of functions f that yield a vanishing collision integral are thus the **Maxwellian distributions**, having the form

$$\begin{aligned} f(\xi) &= \exp\big\{a + b\cdot\xi + c|\xi^2|\big\}\\ &= A\exp\big\{-\beta|\xi - v|^2\big\}\end{aligned} \tag{3.5}$$

for some $A, \beta > 0$, $v \in I\!R^3$. These represent velocity distributions which are in statistical equilibrium.

Returning to the original Boltzmann equation

$$\partial_t f + \xi\cdot\nabla_x f = Q(f), \tag{3.6}$$

multiplying both sides by $\log f$ and integrating w.r.t. ξ we obtain

$$\begin{aligned}\frac{d}{dt}\int f\,\ln f\,d\xi &= \int [f_t\,\ln f + f_t]\,d\xi\\ &= \int Q(f)\,(\ln f + 1)\,d\xi - \int (\xi\cdot\nabla_x f)\,(\ln f + 1)\,d\xi\\ &= \int Q(f)\,\ln f\,d\xi - \int \nabla_x\cdot\big(\xi\,f\,\ln f\big)\,d\xi\,.\end{aligned} \tag{3.7}$$

This yields the macroscopic balance equation

$$\partial_t\mathcal{H}(t,x) + \nabla_x\cdot\mathcal{I}(t,x) = \mathcal{S}(t,x)\,, \tag{3.8}$$

where

$$\mathcal{H} \doteq \int_{I\!R^3} f\,\ln f\,d\xi\,, \qquad \mathcal{I} \doteq \int_{I\!R^3}\xi\,f\,\ln f\,d\xi\,, \qquad \mathcal{S} \doteq \int_{I\!R^3} Q(f)\,\ln f\,d\xi\,.$$

By (3.2), $\mathcal{S} \leq 0$. Hence, assuming suitable decay at $|x| \to \infty$, we obtain

$$\frac{d}{dt}\int_{I\!R^3}\mathcal{H}(t,x)\,dx \leq 0\,.$$

In the homogeneous case where f is independent of x, (3.8) simply yields

$$\frac{d}{dt}\mathcal{H}(t) \leq 0\,.$$

This is Boltzmann's "H-Theorem".

4 - Symmetries

We consider here various transformations that leave invariant the family of solutions to the Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_x f = \int_{I\!R^3}\int_{S^2} (f' f'_* - f f_*)\,\big|\mathbf{n}\cdot(\xi - \xi_*)\big|\,d\mathbf{n}\,d\xi_*\,. \tag{4.1}$$

Let $f = f(t,x,\xi)$ be a solution of (4.1). Then, for every $\bar t \in I\!R$, the function

$$\tilde f(t,x,\xi) \doteq f(t - \bar t,\, x,\, \xi) \qquad \text{(translation in time)} \tag{4.2}$$

provides another solution. The same is true of

$$\tilde f(t,x,\xi) \doteq f(t,\, x - \bar x,\, \xi) \qquad \text{(translation in space)} \tag{4.3}$$

for every $\bar x \in I\!R^3$. Moreover, for every unitary 3×3 matrix U, the function

$$\tilde f(t,x,\xi) \doteq f(t,\, Ux,\, U\xi) \qquad \text{(rotation in space)} \tag{4.4}$$

provides still another solution.

Next, consider a dilation of the form

$$\tilde f = \lambda^\alpha f, \qquad \tilde t = \lambda^\beta t, \qquad \tilde x = \lambda^\gamma x, \qquad \tilde \xi = \lambda^\delta \xi,$$

so that

$$\tilde f(\tilde t, \tilde x, \tilde \xi) \doteq \lambda^\alpha f(\lambda^{-\beta}\tilde t,\, \lambda^{-\gamma}\tilde x,\, \lambda^{-\delta}\tilde \xi), \tag{4.5}$$

with $\lambda > 0$. A direct computation yields

$$\partial_{\tilde t}\tilde f = \lambda^{\alpha-\beta}\,\partial_t f\,,$$

$$\tilde\xi \cdot \nabla_{\tilde x}\tilde f = \lambda^{\delta-\gamma+\alpha}\,\xi\cdot\nabla_x f\,,$$

$$Q(\tilde f) = \int_{I\!R^3}\int_{S^2} (\tilde f' \tilde f'_* - \tilde f \tilde f_*)\,\big|\mathbf{n}\cdot(\tilde\xi - \tilde\xi_*)\big|\,d\mathbf{n}\,d\tilde\xi_* = \lambda^{2\alpha+4\delta}\,Q(f)\,.$$

Therefore, if $f = f(t,x,\xi)$ is a solution, the function

$$\tilde f(t,x,\xi) = \lambda^\alpha\, f(\lambda^{-\beta}t,\, \lambda^{-\gamma}x,\, \lambda^{-\delta}\xi)$$

will be another solution of the Boltzmann equation (4.1) provided that

$$-\beta = \delta - \gamma = \alpha + 4\delta\,. \tag{4.6}$$

We thus have a 2-parameter family of dilations, leaving invariant the solution set of (4.1). In particular, taking $\alpha = 1$, $\beta = \gamma = -1$, $\delta = 0$ we find the new solutions

$$\tilde f(t,x,\xi) = \lambda f(\lambda t,\, \lambda x,\, \xi). \tag{4.7}$$

Moreover, taking $\alpha = 4$, $\beta = 0$, $\gamma = \delta = -1$ we find

$$\tilde f(t,x,\xi) = \lambda^4 f(t,\, \lambda x,\, \lambda\xi). \tag{4.8}$$

5 - The macroscopic balance equations

From the function $f = f(t, x, \xi)$, one can obtain a macroscopic description of the gas, by integration w.r.t. the variable ξ. We thus define the density of **mass** of the gas as

$$\rho(t,x) \doteq \int f(t,x,\xi)\, d\xi\,, \tag{5.1}$$

the density of **momentum** $q = (q_1, q_2, q_3)$,

$$q_i(t,x) \doteq \int \xi_i\, f(t,x,\xi)\, d\xi\,, \tag{5.2}$$

and the **energy** density

$$w(t,x) \doteq \frac{1}{2}\int |\xi|^2\, f(t,x,\xi)\, d\xi\,. \tag{5.3}$$

Moreover, we define the **macroscopic velocity** $v = (v_1, v_2, v_3)$ as

$$v_i \doteq \frac{q_i}{\rho} = \frac{\int \xi_i\, f\, d\xi}{\int f\, d\xi}\,, \tag{5.4}$$

the **momentum flow** $m = m_{ij}$

$$m_{i,j} \doteq \int \xi_i \xi_j\, f\, d\xi \qquad\qquad (i,j = 1,2,3)\,, \tag{5.5}$$

and the **energy flow** $r = (r_1, r_2, r_3)$

$$r_i \doteq \int |\xi|^2 \xi_i\, f\, d\xi\,. \tag{5.6}$$

The microscopic velocity ξ of a particle can now be expressed as a sum

$$\xi = v + c\,,$$

where $c \doteq \xi - v$ represents the random deviation of the velocity of a single particle from the average velocity v. Of course

$$\int c\, f\, d\xi = 0\,.$$

It is convenient to split the momentum flow as

$$m_{ij} = \rho v_i v_j + p_{ij} \tag{5.7}$$

with

$$p_{ij} = \int c_i c_j f\, d\xi\,. \tag{5.8}$$

One can write the energy density as a sum of a **kinetic energy** and an **internal energy**

$$w = \frac{1}{2}\rho|v|^2 + \rho e\,. \tag{5.9}$$

Here e is the **internal energy** per unit mass, defined by

$$\rho e \doteq \frac{1}{2} \int |c|^2 f \, d\xi \,. \tag{5.10}$$

Similarly, the energy flow can be written as

$$r_i = \rho v_i \Big(\frac{1}{2} |v|^2 + e \Big) + \chi_i + \sum_{j=1}^{3} v_j p_{ij} \,, \tag{5.11}$$

where $\chi = (\chi_1, \chi_2, \chi_3)$ is the heat-flow

$$\chi_i \doteq \frac{1}{2} \int |c|^2 c_i \, f \, d\xi \,. \tag{5.12}$$

The evolution of the macroscopic quantities ρ, q, w can be derived from the Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_x f = Q(f), \tag{5.13}$$

multiplying both sides by the collision invariants

$$\phi_0(\xi) \equiv 1 \,, \qquad \phi_i(\xi) = \mathbf{e}_i \cdot \xi_i \quad (i = 1, 2, 3), \qquad \phi_4(\xi) = \frac{|\xi|^2}{2} \,.$$

Observing that

$$\int \phi_j(\xi) \, Q(f) \, d\xi = 0 \qquad\qquad j = 0, 1, 2, 3, 4,$$

from (5.13) we obtain

$$\frac{\partial}{\partial t} \int \phi_j \, f \, d\xi + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int \xi_i \phi_j \, f \, d\xi = 0 \,.$$

In the cases $j = 0$, $j = 1, 2, 3$ and $j = 4$ respectively, one obtains

- **conservation of mass:**

$$\frac{\partial}{\partial t} \rho + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\rho v_i) = 0 \,, \tag{5.14}$$

- **conservation of momentum:**

$$\frac{\partial}{\partial t} (\rho v_j) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\rho v_i v_j + p_{ij}) = 0 \qquad\qquad j = 1, 2, 3, \tag{5.15}$$

- **conservation of energy:**

$$\frac{\partial}{\partial t} \Big(\rho \frac{|v|^2}{2} + \rho e \Big) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[\rho v_i \Big(\frac{|v|^2}{2} + e \Big) + \chi_i + \sum_{j=1}^{3} v_j p_{ij} \right] = 0 \,. \tag{5.16}$$

The above equations are in conservation form, showing that the integral of mass, momentum or energy over a given set Ω can change in time only because of a flux across the boundary $\partial\Omega$. In particular, if this flux vanishes, then the total mass

$$\mathcal{M} \doteq \int_\Omega \rho \, dx \,,$$

the total momentum

$$\mathcal{Q} \doteq \int_\Omega \rho v \, dx \,,$$

and the total energy

$$\mathcal{E} \doteq \int_\Omega \Big(\rho \frac{|v|^2}{2} + \rho e \Big) \, dx$$

inside Ω remain constant in time.

The equations (5.14)-(5.16) represent a system of 5 scalar conservation laws, which however is not closed. Indeed, the flux functions involve the additional quantities p_{ij} and χ_i which depend on higher moments of the distribution $f(t,x,\cdot)$, according to (5.8), (5.12). In order to obtain a hyperbolic system of conservation laws, one can formally proceed as follows. Consider the equation

$$\partial_t f^\epsilon + \xi \cdot \nabla_x f^\epsilon = \frac{1}{\epsilon} Q(f^\epsilon, f^\epsilon). \tag{5.17}$$

When the relaxation parameter $\epsilon > 0$ is very small, we expect that the function f^ϵ will converge pointwise to an equilibrium distribution $\bar f$ such that $Q(\bar f, \bar f) = 0$. According to our previous analysis, this must be a Maxwellian distribution

$$\bar f(t,x,\xi) = A \, \exp\big\{ -\beta |\xi - v|^2 \big\} \,.$$

For each (t,x), the coefficients $A = A(t,x)$, $\beta = \beta(t,x)$ and $v = v(t,x)$ can be uniquely determined by the condition that the densities of mass, momentum and energy in $\bar f(t,x,\cdot)$ should be the same as for $f(t,x,\cdot)$, namely

$$\int_{I\!R^3} \bar f(t,x,\cdot) \, d\xi = \int_{I\!R^3} f(t,x,\cdot) \, d\xi \,,$$

$$\int_{I\!R^3} \xi_i \, \bar f(t,x,\cdot) \, d\xi = \int_{I\!R^3} \xi_i \, f(t,x,\cdot) \, d\xi \qquad\qquad i = 1,2,3 \,,$$

$$\int_{I\!R^3} \frac{|\xi|^2}{2} \, \bar f(t,x,\cdot) \, d\xi = \int_{I\!R^3} \frac{|\xi|^2}{2} \, f(t,x,\cdot) \, d\xi \,.$$

A direct computation yields

$$\beta = \frac{3}{4e} \,, \qquad\qquad A = \rho \left(\frac{4}{3} \pi e \right)^{-3/2} \,,$$

while v is given by (5.4). For the Maxwellian distribution $\bar f$, the stress tensor is diagonal

$$p_{ij} = \begin{cases} \frac{2}{3} \rho e & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \tag{5.18}$$

while the heat flow vector vanishes:

$$\chi_i = 0 \qquad\qquad i = 1,2,3. \tag{5.19}$$

Using (5.18) and (5.19) we can close the system of equations (5.14)-(5.16) and obtain

$$\frac{\partial}{\partial t}\rho + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho v_i) = 0\,, \tag{5.20}$$

$$\frac{\partial}{\partial t}(\rho v_j) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(\rho v_i v_j + \frac{2}{3}\rho e\Big) = 0 \qquad\qquad j = 1,2,3, \tag{5.21}$$

$$\frac{\partial}{\partial t}\Big(\rho \frac{|v|^2}{2} + \rho e\Big) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left[\rho v_i\Big(\frac{|v|^2}{2} + \frac{5}{3}e\Big)\right] = 0\,. \tag{5.22}$$

The equations (5.20)–(5.22) represent a hyperbolic system of five conservation laws in three space dimensions. For this system, an outstanding open problem is to prove the global existence of entropy admissible weak solutions. One conjectures that these solutions can be obtained from solutions to the Boltzmann equation (5.17), letting the relaxation parameter ϵ tend to zero. This would provide a rigorous justification to the formal arguments outlined in this section.

6 - The spatially homogeneous case

In this section we look at the space homogeneous case, where the velocity distribution f does not depend on x. The Cauchy problem thus takes the simpler form

$$\partial_t f = Q(f)\,, \tag{6.1}$$

$$f(0,\xi) = f_0(\xi)\,. \tag{6.2}$$

We will show that the above problem can be reformulated as a continuous differential equation in a suitable Banach space. Global existence and uniqueness of solutions will thus follow from general O.D.E. theory.

It will convenient to work in a space a weighted integrable functions, with norm

$$\|f\|_{1,s} \doteq \int (1+\xi^2)^{s/2}\big|f(\xi)\big|\,d\xi\,. \tag{6.3}$$

For $s \geq 2$ we thus define the space of velocity distributions having moments up to order s:

$$\mathbf{L}^1_s \doteq \big\{f : I\!R^3 \mapsto I\!R\,, \quad \|f\|_{1,s} < \infty\big\}\,.$$

Here and in the sequel, with slight abuse of notation we write $\xi^2 \doteq |\xi|^2$. For future use, we record the interpolation inequality

$$\|h\|_{1,3} \leq \sqrt{\|h\|_{1,4}\cdot\|h\|_{1,2}}\,. \tag{6.4}$$

To prove (6.4), we can assume $h \geq 0$ and consider the absolutely continuous measure such that $d\mu = h(\xi)d\xi$. Using Cauchy's inequality one obtains

$$\begin{aligned}
\|h\|_{1,3} &= \int (1+\xi^2)^{2/3}\, h(\xi)\, d\xi \;=\; \int (1+\xi^2)\cdot(1+\xi^2)^{1/2}\, d\mu \\
&\leq \|1+\xi^2\|_{\mathbf{L}^2(\mu)} \cdot \big\|(1+\xi^2)^{1/2}\big\|_{\mathbf{L}^2(\mu)} \\
&= \left(\int (1+\xi^2)^2\, h(\xi)\, d\xi\right)^{1/2} \cdot \left(\int (1+\xi^2)\, h(\xi)\, d\xi\right)^{1/2} \\
&= \|h\|_{1,4}^{1/2} \cdot \|h\|_{1,2}^{1/2}\,.
\end{aligned}$$

Theorem 6.1. *Assume that $f_0 \in L^1_4$ and $f_0(\xi) \geq 0$ for a.e. ξ. Then the Cauchy problem (6.1)-(6.2) has a unique solution, defined for all $t \geq 0$. The map $t \mapsto f(t)$ is continuously differentiable as a map with values in $\mathbf{L}^1_2$. Moreover, for every $t \geq 0$ there holds*

$$\int f(t,\xi)\, d\xi = \int f_0(\xi)\, d\xi\,, \qquad \int \left(1+\xi^2\right) f(t,\xi)\, d\xi = \int \left(1+\xi^2\right) f_0(\xi)\, d\xi\,. \tag{6.5}$$

Toward a proof we observe that, by a possible rescaling of the time variable, it is not restrictive to assume the normalization

$$\int f(\xi)\, d\xi = 1\,. \tag{6.6}$$

On the space $\mathbf{L}^1_2$ we consider a closed convex subset Ω of the form

$$\begin{aligned}
\Omega = \Bigg\{ f \in \mathbf{L}^1_2\,; \qquad & f(\xi) \geq 0 \;\text{ for a.e. } \xi\,, \qquad \int f(\xi)\, d\xi = 1\,, \\
& \int \left(1+|\xi|^2\right) f(\xi)\, d\xi \leq \kappa_2\,, \qquad \int \left(1+|\xi|^2\right)^2 f(\xi)\, d\xi \leq \kappa_4 \Bigg\}
\end{aligned} \tag{6.7}$$

for some constants κ_2, κ_4 In order to apply a general theorem on O.D.E's in Banach spaces, we need to show that the collision operator $f \mapsto Q(f)$, as a map from Ω into L^1_2, satisfies the following::

- Hölder continuity.
- One-sided Lipschitz condition.
- Tangency condition.

We begin by proving some estimates on the collision kernel, in the norms $\|\cdot\|_{1,s}$.

Lemma 6.2. *The map $f \mapsto Q(f)$ is uniformly Hölder continuous on $\mathbf{L}^1_2$, when restricted to the domain Ω.*

Proof. We shall consider the loss term Q_- and the gain term Q_+ separately. Let $f, g \in \Omega$. Observing that

$$\big|\mathbf{n}\cdot(\xi-\xi_*)\big| \leq |\xi-\xi_*| \leq |\xi| + |\xi_*| = \left(|\xi|^2 + |\xi_*|^2 + 2|\xi||\xi_*|\right)^{1/2} \leq (1+\xi^2)^{1/2}(1+\xi_*^2)^{1/2}, \tag{6.8}$$

and using the interpolation inequality (6.4) we compute

$$
\begin{aligned}
\big\|Q_-(f)-Q_-(g,g)\big\|_{1,2} &= \int_{I\!R^3} (1+\xi^2) \int_{I\!R^3}\int_{S^2} \big|\mathbf{n}\cdot(\xi-\xi_*)\big|\,|ff_*-gg_*|\,d\mathbf{n}\,d\xi_*\,d\xi \\
&\le 2\pi \iint (1+\xi^2)^{3/2}(1+\xi_*^2)^{1/2}\cdot\Big(\big|f(\xi)-g(\xi)\big|\,f(\xi_*)+g(\xi)\big|f(\xi_*)-g(\xi_*)\big|\Big)\,d\xi d\xi_* \\
&= 2\pi\Big(\big\|f-g\big\|_{1,3}\|f\|_{1,2}+\big\|f-g\big\|_{1,2}\|g\|_{1,3}\Big) \\
&\le 2\pi\Big(\big\|f-g\big\|_{1,4}^{1/2}\|f-g\|_{1,2}^{1/2}\|f\|_{1,2}+\big\|f-g\big\|_{1,2}\|g\|_{1,3}\Big) \\
&\le 2\pi\Big((2\kappa_2)^{1/2}\kappa_2\cdot\|f-g\|_{1,2}^{1/2}+\kappa_4^{1/2}\kappa_2^{1/2}\cdot\|f-g\|_{1,2}\Big)\,.
\end{aligned}
\tag{6.9}
$$

The estimate for the gain terms is entirely similar:

$$
\begin{aligned}
\big\|Q_+(f)-Q_+(g,g)\big\|_{1,2} &= \int_{I\!R^3} (1+\xi^2) \int_{I\!R^3}\int_{S^2} \big|\mathbf{n}\cdot(\xi-\xi_*)\big|\,|f'f'_*-g'g'_*|\,d\mathbf{n}\,d\xi_*\,d\xi \\
&\le \int_{I\!R^3}\int_{I\!R^3}\int_{S^2} \big[(1+\xi'^2)+(1+{\xi'_*}^2)\big]\,\big|\mathbf{n}\cdot(\xi'-\xi'_*)\big|\,|f'f'_*-g'g'_*|\,d\mathbf{n}\,d\xi'_*\,d\xi' \\
&\le 2\iiint (1+\xi^2)\big|\mathbf{n}\cdot(\xi-\xi_*)\big|\,|ff_*-gg_*|\,d\mathbf{n}\,d\xi_*\,d\xi \\
&\le 4\pi\Big((2\kappa_2)^{1/2}\kappa_2\cdot\|f-g\|_{1,2}^{1/2}+\kappa_4^{1/2}\kappa_2^{1/2}\cdot\|f-g\|_{1,2}\Big)\,.
\end{aligned}
\tag{6.10}
$$

We have here used the symmetry of the collision kernel Q_+ w.r.t. the variables ξ,ξ_*. Together, (6.9) and (6.10) establish the lemma. □

We remark that, taking $g=0$, the above computations yield

$$
\big\|Q(f)\big\|_{1,2} \;\le\; 6\pi\,\|f\|_{1,3}\,\|f\|_{1,2} \;\le\; 6\pi\,\kappa_4^{1/2}\,\kappa_2^{3/2}. \tag{6.11}
$$

Next, we show that the collision operator Q satisfies a one-sided Lipschitz estimate. Consider the following semi-inner product on the space L^1_2 :

$$
\langle f,w\rangle_{1,2} \doteq \|f\|_{1,2}\cdot[f,w]_{1,2}\,,
$$

where

$$
[f,w]_{1,2} \doteq \int (1+\xi^2)\,\mathrm{sign} f(\xi)\cdot w(\xi)\,d\xi\,.
$$

We recall that the collision operator Q replaces two particles with speeds ξ,ξ_* with a cloud of particles with speeds ξ',ξ'_* uniformly distributed on the sphere having $\xi\xi_*$ as diameter. In the following computation we use the symmetry of the integral expressing $Q(f)$ w.r.t. the variables ξ and ξ_*. Moreover, we use and the identity $\xi^2+\xi_*^2={\xi'}^2+{\xi'_*}^2$ and the upper bound (6.8) for the distance $|\xi - \xi_*|$. This yields

$$
\begin{aligned}
&\big[f-g\,,\ Q(f)-Q(g)\big]_{1,2} \\
&\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \big|\mathbf{n}\cdot(\xi-\xi_*)\big|\cdot\bigg\{\big((1+\xi'^2)+(1+\xi_*'^2)\big)\cdot\big|f(\xi)f(\xi_*)-g(\xi)g(\xi_*)\big| \\
&\quad -\Big[(1+\xi^2)\,\mathrm{sign}\big(f(\xi)-g(\xi)\big)+(1+\xi_*^2)\,\mathrm{sign}\big(f(\xi_*)-g(\xi_*)\big)\Big]\,\big(f(\xi)f(\xi_*)-g(\xi)g(\xi_*)\big)\bigg\}\,d\mathbf{n}\,d\xi\,d\xi_* \\
&= 2\pi\iint|\xi-\xi_*|\cdot\bigg\{\big((1+\xi^2)+(1+\xi_*^2)\big)\cdot\frac{1}{2}\Big|(f+g)(\xi)(f-g)(\xi_*)+(f+g)(\xi_*)(f-g)(\xi)\Big| \\
&\qquad -\Big[(1+\xi^2)\,\mathrm{sign}\big(f(\xi)-g(\xi)\big)+(1+\xi_*^2)\,\mathrm{sign}\big(f(\xi_*)-g(\xi_*)\big)\Big] \\
&\qquad\qquad \cdot\frac{1}{2}\Big[(f+g)(\xi)(f-g)(\xi_*)+(f+g)(\xi_*)(f-g)(\xi)\Big]\bigg\}\,d\xi\,d\xi_* \\
&= 2\pi\iint|\xi-\xi_*|\cdot\bigg\{(1+\xi^2)\cdot\Big|(f+g)(\xi)(f-g)(\xi_*)+(f+g)(\xi_*)(f-g)(\xi)\Big| \\
&\qquad -(1+\xi^2)\,\mathrm{sign}\big(f(\xi)-g(\xi)\big)\cdot\Big[(f+g)(\xi)(f-g)(\xi_*)+(f+g)(\xi_*)(f-g)(\xi)\Big]\bigg\}\,d\xi\,d\xi_* \\
&\leq 2\pi\iint|\xi-\xi_*|\cdot(1+\xi^2)\cdot\bigg\{\Big[(f+g)(\xi)|f-g|(\xi_*)+(f+g)(\xi_*)|f-g|(\xi)\Big] \\
&\qquad +\Big[(f+g)(\xi)|f-g|(\xi_*)-(f+g)(\xi_*)|f-g|(\xi)\Big]\bigg\}\,d\xi\,d\xi_* \\
&\leq 4\pi\iint(1+\xi^2)^{1/2}(1+\xi_*^2)^{1/2}\cdot(1+\xi^2)\cdot(f+g)(\xi)(f-g)(\xi_*)\,d\xi\,d\xi_* \\
&= 4\pi\,\|f+g\|_{1,3}\cdot\|f-g\|_{1,2} \\
&\leq 8\pi\kappa_4\cdot\|f-g\|_{1,2}\,.
\end{aligned}
\tag{6.12}
$$

We now consider the tangency condition. Since the collision operator preserves total mass and energy, for every $f\in\Omega$ and $h>0$ we have

$$
\int\Big(f(\xi)+h\,Q(f)(\xi)\Big)\,d\xi=1\,,\qquad \int(1+\xi^2)\cdot\Big(f(\xi)+h\,Q(f)(\xi)\Big)\,d\xi=\int(1+\xi^2)\cdot f(\xi)\,d\xi\leq\kappa_2\,.
\tag{6.13}
$$

Moreover, the vector $Q(f)$ is tangent to the positive cone $\Gamma_+\doteq\big\{f\in\mathbf{L}^1_2;\quad f(\xi)\geq 0\big\}$ simply because $Q(f)(\xi)\geq 0$ whenever $f(\xi)=0$.

The forthcoming analysis will establish the a priori bound $\|f\|_{1,4}\leq\kappa_4$, provided that the constant κ_4 is suitably large. This will require further estimates on the collision kernel.

Lemma 6.3 (Povzner). *For every $f\in\mathbf{L}^1_s$ with $f\geq 0$ and $s\geq 2$, one has the estimate*

$$
\int\big(1+|\xi|^2\big)^{s/2}Q(f)\,d\xi\leq C_s\cdot\|f\|_{1,s}\|f\|_{1,2}
\tag{6.14}
$$

for some constant C_s, with $C_s\to 0$ as $s\to 2$.

Proof. We first observe that, for any $p\geq 1$ there exists a constant C'_p such that

$$
a^p+b^p\leq(a+b)^p\leq a^p+b^p+C'_p(a^{1/2}b^{p-(1/2)}+a^{p-(1/2)}b^{1/2})
\tag{6.15}
$$

for any numbers $a, b \geq 0$. Moreover, $C'_p \to 0$ as $p \to 1$.

If now ξ, ξ_*, ξ', ξ'_* are related as in (1.1), then by (6.15)

$$\begin{aligned}(1+\xi'^2)^{s/2} + (1+\xi'^2_*)^{s/2} \ &\leq \ \left[1+\xi'^2+1+\xi'^2_*\right]^{s/2} \ = \ \left[1+\xi^2+1+\xi^2_*\right]^{s/2} \\ &\leq (1+\xi^2)^{s/2} + (1+\xi^2_*)^{s/2} + C'_{s/2} \cdot \Big((1+\xi^2)^{1/2}(1+\xi^2_*)^{(s-1)/2} + (1+\xi^2)^{(s-1)/2}(1+\xi^2_*)^{1/2}\Big)\end{aligned} \tag{6.16}$$

By the action of the collision kernel Q, a couple of particles with speeds ξ, ξ_* is replaced by particles whose speeds are uniformly distributed on the sphere $S_{\xi\xi_*}$. Using (6.8) and then (6.16) to cancel the leading order terms, we thus obtain

$$\begin{aligned}\int \left(1+|\xi|^2\right)^{s/2} Q(f)\,d\xi &\leq \int\!\!\int\!\!\int \left|\mathbf{n}\cdot(\xi-\xi_*)\right| f(\xi)f(\xi_*) \\ &\qquad \cdot \Big|(1+\xi'^2)^{s/2} + (1+\xi'^2_*)^{s/2} - (1+\xi^2)^{s/2} - (1+\xi^2_*)^{s/2}\Big|\,d\mathbf{n}\,d\xi_* d\xi \\ &\leq \pi \int\!\!\int (1+\xi^2)^{1/2}(1+\xi^2_*)^{1/2}\ f(\xi)f(\xi_*) \\ &\qquad \cdot C'_{s/2}\left[(1+\xi^2)^{1/2}(1+\xi^2_*)^{(s-1)/2} + (1+\xi^2)^{(s-1)/2}(1+\xi^2_*)^{1/2}\right] d\xi_* d\xi \\ &\leq C_s \cdot \|f\|_{1,s}\|f\|_{1,2}\,,\end{aligned}$$

with $C_s = 2\pi\, C'_{s/2}$. □

The estimate (6.14) provides an a-priori bound on the $\mathbf{L}^1_s$ norm of a solution. Namely, by mass and energy conservation there holds

$$\big\|f(t)\big\|_{\mathbf{L}^1} = \big\|f_0\big\|_{\mathbf{L}^1} \qquad\qquad \big\|f(t)\big\|_{1,2} = \big\|f_0\big\|_{1,2} \tag{6.17}$$

for all $t \geq 0$. In turn, (6.14) implies

$$\frac{d}{dt}\big\|f(t)\big\|_{1,s} \leq C_s\|f_0\|_{1,2}\cdot\big\|f(t)\big\|_{1,s}, \qquad\qquad \big\|f(t)\big\|_{1,s} \leq \exp\left\{C_s\|f_0\|_{1,2}\cdot t\right\}\cdot\|f_0\|_{1,s}\,. \tag{6.18}$$

A sharper estimate is now given. First, consider the case of a particle with speed 0 colliding with a particle with speed ξ (each particle having unit mass). The result is a cloud of particles (with total mass = 2) uniformly distributed along the sphere $S_{0\xi}$, having a diameter joining the points 0 with ξ, and radius $R = |\xi|/2$. Setting

$$r(\theta) = 2R\cos\theta\,, \qquad\qquad x(\theta) = R\,(1+\cos 2\theta)\,,$$

we now compute the momentum of order s of this cloud of particles (fig. 7)

$$\begin{aligned} I_s &\doteq \frac{2}{4\pi R^2}\int_{S_{0\xi}} |\xi'|^s\,d\sigma \\ &= \frac{1}{2\pi R^2}\int_0^{\pi/2} \big[r(\theta)\big]^s \cdot 2\pi R\,\big|dx(\theta)\big| \\ &= \frac{1}{2\pi R^2}\int_0^{\pi/2} (2R\cos\theta)^s \cdot 2\pi R\cdot 2R\sin 2\theta\,d\theta \\ &= (2R)^s\int_0^{\pi/2} (4\cos\theta)^{s+1}\sin\theta\,d\theta \\ &= |\xi|^s\cdot\frac{4}{s+2}\,. \end{aligned} \tag{6.19}$$

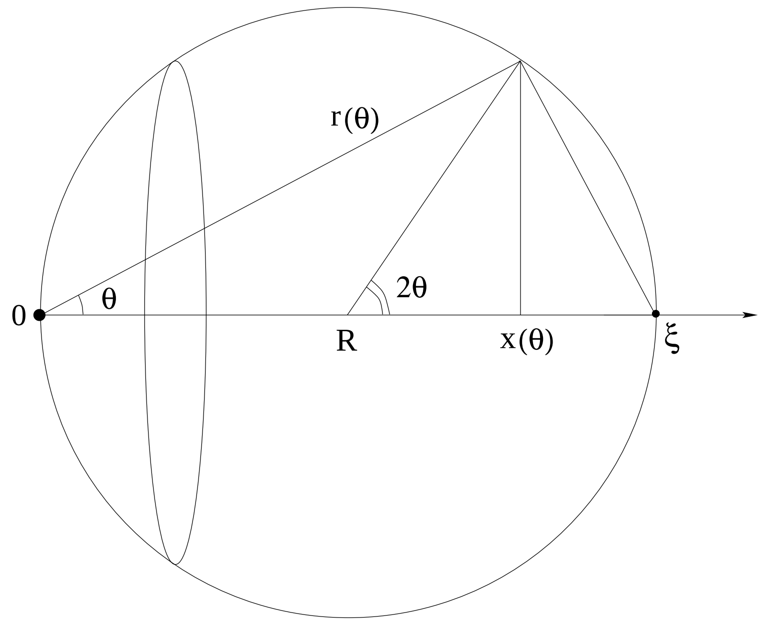

figure 7

Notice that when $s = 2$ we have $I_s = |\xi|^s$. Otherwise

$$I_s < |\xi|^s \qquad\qquad s > 2\,.$$

This reflects the fact that collisions conserve the energy norm, but dissipate higher norms L^1_s with $s > 2$. By a simple rescaling argument we obtain

Lemma 6.4. *For each $s > 2$ and any $\lambda \in \,\big]4/(s+2)\,,\; 1\big[$, one can find a constant α large enough so that the following holds. If $|\xi| \geq \alpha\,\big(1 + |\xi_*|\big)$, then*

$$\frac{2}{\pi\,|\xi - \xi_*|^2}\int_{S_{\xi\xi_*}} (1 + |\xi'|^2)^{s/2}\,d\sigma \leq \lambda\left[(1+\xi^2)^{s/2} + (1+\xi_*^2)^{s/2}\right], \tag{6.20}$$

where $d\sigma$ denotes the surface area on $S_{\xi\xi_*}$.

Proof. If the conclusion of the lemma fails, then there exists a sequence of couples ξ^ν, ξ^ν_* with

$$|\xi^\nu_*| \geq 1\,, \qquad\qquad \frac{|\xi^\nu|}{|\xi^\nu_*|} \to \infty\,,$$

such that (6.20) fails for every ν. Set

$$\zeta^\nu \doteq \frac{2\xi^\nu}{|\xi^\nu|}\,, \qquad\qquad \zeta^\nu_* \doteq \frac{2\xi^\nu_*}{|\xi^\nu|}\,.$$

As $\nu \to \infty$, by possibly taking a subsequence we can assume that ζ^ν converges to some vector ζ. Clearly $\zeta^\nu_* \to 0$. Hence the sphere whose diameter is the segment joining ζ^ν_* with ζ^ν approaches a sphere with unit radius, whose diameter is the segment with endpoints 0, ζ. Using (6.9) with $R = 1$ we thus find

$$\lim_{\nu\to\infty} \frac{1}{(1+|\xi^\nu|^2)^{s/2}} \cdot \frac{2}{\pi\,|\xi^\nu - \xi^\nu_*|^2} \int_{S_{\xi^\nu \xi^\nu_*}} \big(1+|\xi'|^2\big)^{s/2}\, d\sigma$$
$$= \frac{1}{|\zeta|^s} \cdot \frac{2}{\pi\,|\zeta|^2} \int_{S_{0\zeta}} |\zeta'|^s\, d\sigma = \frac{4}{s+2} > \lambda\,.$$

This contradiction establishes the lemma. □

Lemma 6.5. *Let $t \mapsto f(t)$ be a solution of the homogeneous Boltzmann equation (6.1), with $\big\|f(t)\big\|_{1,2} \le \kappa_2$ and $\big\|f(t)\big\|_{1,s}$ locally bounded, for some $s > 2$. Then there exist a constant κ_s depending only on s and κ_2 such that*

$$\frac{d}{dt}\,\big\|f(t)\big\|_{1,s} \le 0 \qquad \textit{whenever} \quad \|f(t)\|_{1,s} \ge \kappa_s\,. \tag{6.21}$$

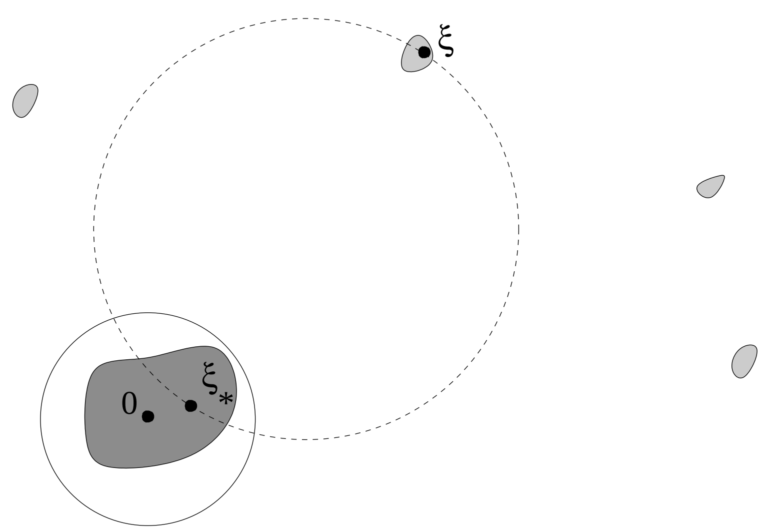

figure 8

The situation is illustrated in fig. 8. If the norm $\|f\|_{1,2}$ is small, most particles have small speed, contained in a neighborhood V of the origin. However, if the norm $\|f\|_{1,s}$ is very large, there must be a few particles having very large speed. By the interactions between these high speed particles and other slow particles, the $\mathbf{L}^1_s$ norm is dissipated, as shown in Lemma 6.4.

Proof of Lemma 6.5. If $t \mapsto f(t)$ is a solution, its $\mathbf{L}^1_2$ norm is clearly constant in time. Assume $\|f\|_{1,2} \le \kappa_2$. For a given $s > 2$, choose constants $\lambda < 1$ and α according to Lemma 6.4. Define the radius

$$\rho \doteq \sqrt{2\kappa_2}\,.$$

Recalling the normalization $\int f(\xi)d\xi = 1$, this implies

$$\int_{|\xi_*|\le\rho} f(\xi_*)\, d\xi_* > \frac{1}{2}\,.$$

In the integral expression for $Q(f)$, we now split the domain $I\!R^3 \times I\!R^3 = \Gamma \cup \Gamma'$, setting

$$\Gamma' \doteq \big\{(\xi, \xi_*)\,;\quad \xi_* \leq \rho\,,\ \xi \geq \alpha(1+\rho)\big\}\,, \qquad\qquad \Gamma \doteq I\!R^3 \times I\!R^3 \setminus \Gamma'\,.$$

Using the two previous lemmas we find

$$\begin{aligned}
\frac{d}{dt}\|f(t)\|_{1,s} &= \iint_{\Gamma\cup\Gamma'} \int_{S^2} \big|\mathbf{n}\cdot(\xi-\xi_*)\big|\, f(\xi) f(\xi_*) \\
&\qquad \cdot \Big[(1+\xi'^2)^{s/2} + (1+\xi_*'^{\,2})^{s/2} - (1+\xi^2)^{s/2} - (1+\xi_*^2)^{s/2}\Big]\, d\mathbf{n}\, d\xi_* d\xi \\
&\leq C_s \cdot \big\|f(t)\big\|_{1.2}\big\|f(t)\big\|_{1,s} - (1-\lambda)\iint_{\Gamma'} |\xi-\xi_*|(1+\xi^2)^{s/2} f(\xi) f(\xi_*)\, d\xi d\xi_* \\
&\leq C_s \cdot \big\|f(t)\big\|_{1.2}\big\|f(t)\big\|_{1,s} - \frac{1-\lambda}{2}\int_{|\xi|\geq\alpha(1+\rho)} \frac{(1+\xi^2)^{1/2}}{2}(1+\xi^2)^{s/2} f(\xi)\, d\xi\,.
\end{aligned} \tag{6.22}$$

We now apply Jensen's inequality

$$\Psi\left(\int_X u\, d\mu\right) \leq \int_X \Psi(u)\, d\mu$$

to the convex function $\Psi(y) = y^{(s+1)/s}$ and to the probability measure $d\mu = f(\xi)d\xi$, with $u(\xi) = (1+\xi^2)^{s/2}$. This yields

$$\begin{aligned}
\left(\int (1+\xi^2)^{s/2} f(\xi)\, d\xi\right)^{(s+1)/s} &\leq \int (1+\xi^2)^{(s+1)/2} f(\xi)\, d\xi \\
&\leq \int_{|\xi|\geq\alpha(1+\rho)} (1+\xi^2)^{(s+1)/2} f(\xi)\, d\xi + \big(1+\alpha^2(1+\rho)^2\big)^{(s+1)/2}\,.
\end{aligned} \tag{6.23}$$

Using (6.23) to provide a lower bound on the negative term on the right hand side of (6.22), we obtain

$$\frac{d}{dt}\,\|f\|_{1,s} \leq C_s \cdot \|f\|_{1,2}\|f\|_{1,s} - \frac{1-\lambda}{4}\,\|f\|_{1,s}^{(s+1)/s} + \frac{1-\lambda}{4}\,\big(1+2\alpha^2(1+\rho^2)\big)^{(s+1)/2}. \tag{6.24}$$

The left hand side of (6.24) is ≤ 0 provided that $\|f\|_{1,s}$ is large enough. □

Recalling that $\lambda < 1$ and $s > 2$, the equation (6.24) can be compared with the O.D.E.

$$\dot z = -a\, z^{(s+1)/s} + bz + c \tag{6.25}$$

with

$$a = \frac{1-\lambda}{4}\,, \qquad\qquad b = C_s\, \|f\|_{1,2}\,, \qquad\qquad c = \Big(1+2\alpha^2(1+\sqrt{2\kappa_2})^2\Big)^{(s+1)/2}.$$

For z large, the right hand side of (6.25) is $< -(a/2)z^{(s+1)/s}$. By a comparison with the O.D.E.

$$\dot z = -\frac{a}{2}\, z^{(s+1)/s},$$

we conclude that every solution will satisfy

$$\big\|f(t)\big\|_{1,s} \leq K \cdot (1+t^{-s}) \qquad\qquad s > 2\,,\quad t > 0\,, \tag{6.26}$$

where the constant $K = K\big(s,\ \|f_0\|_{1,2}\big)$ depends only on s and on the $\mathbf{L}^1_2$ norm of the initial data.

Corollary 6.6. *For any given κ_2, there exists a constant κ_4 such that the following holds. If*

$$\|f\|_{1,2} \leq \kappa_2\,, \qquad\qquad \|f\|_{1,4} \geq \kappa_4\,,$$

then

$$\int (1+\xi^2)^2\, Q(f)(\xi)\, d\xi \leq 0\,. \tag{6.27}$$

Thanks to the above estimates, we can now establish Theorem 2 by applying a general existence theorem for O.D.E's in a Banach space. Indeed, we have shown that the map $f \mapsto Q(f)$ is uniformly bounded and Hölder continuous on the closed convex set $\Omega \subset \mathbf{L}^1_2$. Moreover, it satisfies the one-sided Lipschitz condition

$$\big\langle f-g\,,\ Q(f)-Q(g)\big\rangle_{1,2} \doteq \|f-g\|_{1,2}\cdot\big[f-g\,,\ Q(f)-Q(g)\big]_{1,2} \leq 8\pi\kappa_4\cdot\|f-g\|^2_{1,2}\,,$$

together with the tangency condition

$$\lim_{h\to 0+}\frac{1}{h}\,\text{dist.}\big(f+h\,Q(f)\,;\ \Omega\big) = 0 \qquad\qquad f\in\Omega\,.$$

By applying Theorem A1 in the Appendix, we now obtain the global existence of a unique solution to the Cauchy problem (6.1)-(6.2). □

7 - Pointwise bounds

In this chapter we consider the case of bounded initial data. Our goal is to establish uniform $\mathbf{L}^\infty$ bounds valid for all times. We always assume that the density is normalized as in (6.6).

Theorem 7.1. *In the same setting as Theorem 6.1, assume that the initial data is bounded and has finite entropy, so that*

$$\|f_0\|_{\mathbf{L}^\infty} < \infty\,, \qquad\qquad H_0 \doteq \int f_0(\xi)\,\ln f_0(\xi)\,d\xi < \infty\,. \tag{7.1}$$

Moreover, assume that

$$\sup_E \int_E f(\xi)\,d\xi < \infty\,,$$

the supremum being taken over all two-dimensional planes $E\subset I\!R^3$. Then the solution remains uniformly bounded, i.e. $\big\|f(t,\cdot)\big\|_{\mathbf{L}^\infty} \leq C$ for all $t\geq 0$.

Proof. We proceed in several steps.

1. We claim that there exists a constant $c_0>0$ depending only on the entropy H_0, such that

$$\int |\xi_*-\xi|\,f(\xi_*)\,d\xi_* \geq c_0 \tag{7.2}$$

for all $\xi \in I\!R^3$, $t \geq 0$. Indeed, we can find $r_1 > 0$ sufficiently small such that $r \in [0, r_1]$ and

$$\int_{|\xi_* - \xi| \leq r} f(\xi_*)\, d\xi_* \geq \frac{1}{2}$$

together imply

$$\int_{|\xi_* - \xi| \leq r, \, , f(\xi_*) > r^{-1}} f(\xi_*)\, d\xi_* \geq \frac{1}{4}\,. \tag{7.3}$$

If (7.3) holds, then

$$H_0 \geq \int f(\xi_*) \, \ln f(\xi_*)\, d\xi_* \geq \int_{|\xi_* - \xi| \leq r, \, f(\xi_*) > r^{-1}} f(\xi_*) \, |\ln r| \, d\xi_* \geq \frac{|\ln r|}{4}\,.$$

Therefore, setting $r_0 \doteq \min\left\{r_1\,,\ e^{-4H_0}\right\}$, for every $\xi \in I\!R^3$ the above estimates yield

$$\int_{|\xi_* - \xi| \leq r_0} f(\xi_*)\, d\xi_* \leq \frac{1}{2}\,,$$

$$\int |\xi_* - \xi| \, f(\xi_*)\, d\xi_* \geq \int_{|\xi_* - \xi| > r_0} |\xi_* - \xi| \, f(\xi_*)\, d\xi_* \geq \frac{r_0}{2}\,,$$

proving (7.2).

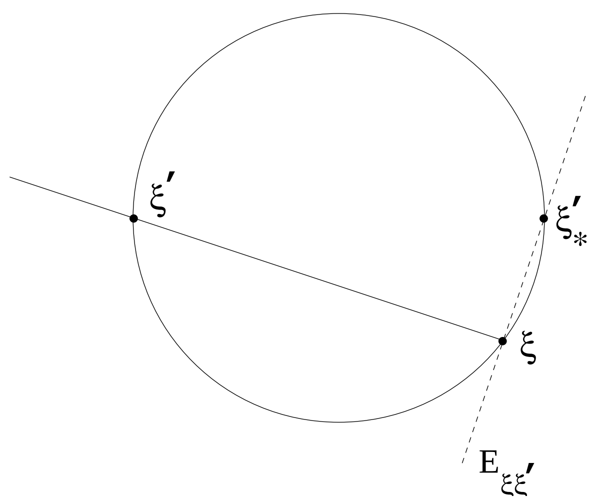

figure 9

2. We shall need an alternative expression for the gain term (fig. 9). In order to produce a particle with speed ξ, one can first choose a particle with arbitrary speed ξ'. This has to interact with some other particle whose speed ξ'_* lies in the plane $E_{\xi\xi'}$ through ξ perpendicular to $\xi' - \xi$. This motivates the formula

$$Q_+(f)(\xi) = \int \frac{1}{|\xi' - \xi|} \left(\int_{E_{\xi\xi'}} f(\xi'_*)\, d\xi'_* \right) f(\xi')\, d\xi'\,. \tag{7.4}$$

The factor $|\xi' - \xi|^{-1}$ comes from (1.6) through a change of variables. A rigorous justification of (7.4) requires some care, because one cannot regard $\xi' \in I\!R^3$ and $\xi'_* \in I\!R^2$ as independent variables. Indeed, the range of $\xi'_* \in E_{\xi\xi'}$ itself depends on ξ'.

To establish (7.4) we thus proceed as follows. Fix $\xi^\dagger \in I\!R^3$ and let ξ'_* range in the fixed plane $E_{\xi\xi^\dagger}$ passing through ξ and perpendicular to $\xi^\dagger - \xi$ (see fig. 10a). For each triple $(s, \mathbf{n}, \tilde{\xi}) \in I\!R^+ \times S^2 \times E_{\xi\xi^\dagger}$ we consider the two speeds

$$\xi' \doteq s\,\mathbf{n}\,, \qquad\qquad \xi'_* = \pi_E \tilde{\xi}\,, \tag{7.5}$$

where π_E denotes the perpendicular projection on the plane $E_{\xi\xi'}$. When $\xi' = \xi^\dagger$, the determinant of the map $(s, \mathbf{n}, \tilde{\xi}) \mapsto (\xi', \xi'_*)$ defined at (7.5) is computed as (see fig. 10b)

$$\det\left(\frac{\partial(\xi',\,\xi'_*))}{\partial(s,\,\mathbf{n},\,\tilde{\xi})}\right) = \det\left(\frac{\partial(s\,\mathbf{n},\ \pi_E\tilde{\xi})}{\partial(s,\,\mathbf{n},\,\tilde{\xi})}\right) = \det\left(\frac{\partial(s\mathbf{n})}{\partial(s,\,\mathbf{n})}\right) = s^2\,.$$

Therefore, from (1.6) (with $\alpha = 1$) it follows

$$Q_+(f)(\xi) = \int_{I\!R^3}\int_{E_{\xi\xi'}} s\cdot f(\xi'_*)\,f(\xi')\cdot\frac{1}{s^2}\,d\xi'_*\,d\xi',$$

proving (7.4).

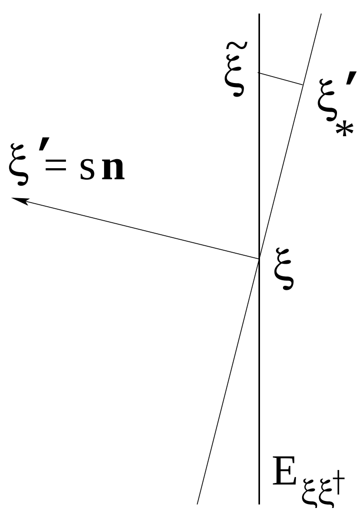

figure 10a

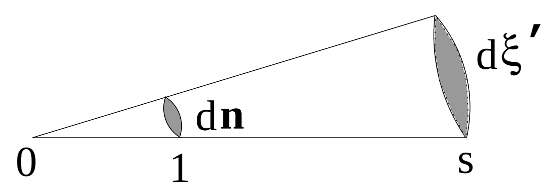

figure 10b

We now consider the quantities

$$J(\xi) \doteq \int \frac{\ }{|\xi' - \xi|}\,f(\xi')\,d\xi', \qquad\qquad I(E) \doteq \int_E f(\xi)\,d\xi\,.$$

From (7.4) it follows

$$Q_+(f)(\xi) \le J(\xi)\cdot\sup_E\,I(E)\,, \tag{7.6}$$

where the supremum is taken over all planes $E \subset I\!R^3$.

3. For any plane E and $\delta > 0$, denote by E_δ the set of points whose distance from E is $\leq \delta$. As usual, $S_{\xi\xi_*}$ is the sphere having the segment $\xi\xi_*$ as diameter. Using (7.2) we compute

$$\begin{aligned}
\frac{d}{dt} I(E) &= \int_E \big[Q_+(f)(\xi) - Q_-(f)(\xi)\big]\, d\xi \\
&\leq \iint \frac{f(\xi)f(\xi_*)}{|\xi - \xi_*|} \left[\lim_{\delta\to 0} \frac{1}{2\delta} \cdot \mathrm{meas}\big(S_{\xi\xi_*} \cap E_\delta\big)\right] d\xi d\xi_* - 2\pi \int_E f(\xi) \int |\xi_* - \xi|\, f(\xi_*)\, d\xi_* d\xi \\
&\leq \pi \iint f(\xi) f(\xi_*)\, d\xi d\xi_* - 2\pi c_0 \int_E f(\xi) d\xi \\
&\leq \pi\big(1 - 2c_0 I(E)\big)\,.
\end{aligned} \tag{7.7}$$

Indeed, the area of the portion of the sphere $S_{\xi\xi_*}$ which is contained inside E_δ satisfies

$$\mathrm{meas}\big(S_{\xi\xi_*} \cap E_\delta\big) \leq 2\delta \cdot 2\pi R\,, \qquad\qquad R \doteq \frac{|\xi - \xi_*|}{2}\,.$$

4. Setting $\xi_0 \doteq (\xi_1 + \xi_2)/2$ and writing $d\sigma$ for the element of area on the sphere $S_{\xi_1\xi_2}$, recalling (7.2) we compute

$$\begin{aligned}
\frac{d}{dt} J(\xi) &= \int \frac{Q_+(f)(\xi') - Q_-(f)(\xi')}{|\xi' - \xi|}\, d\xi' \\
&\leq \iint \frac{1}{|\xi_1 - \xi_2|} \left(\int_{S_{\xi_1\xi_2}} \frac{1}{|\xi' - \xi|}\, d\sigma(\xi') \right) f(\xi_1) f(\xi_2)\, d\xi_1 d\xi_2 \\
&\qquad - 2\pi \int \frac{f(\xi')}{|\xi' - \xi|}\, d\xi' \cdot \int |\xi_* - \xi|\, f(\xi_*) d\xi_* \\
&\leq \iint \frac{1}{|\xi_1 - \xi_2|} \left(\int_{S_{\xi_1\xi_2}} \frac{1}{|\xi' - \xi_0|}\, d\sigma(\xi') \right) f(\xi_1) f(\xi_2)\, d\xi_1 d\xi_2 - 2\pi\, J(\xi)\, c_0 \\
&= 4\pi - 2\pi c_0 J(\xi)\,.
\end{aligned} \tag{7.8}$$

5. By (7.7) and (7.8), the quantities

$$\widehat{J} \doteq \sup_\xi J(\xi)\,, \qquad\qquad \widehat{I} \doteq \sup_E I(E)\,,$$

remain uniformly bounded for all times $t \geq 0$. By (7.6) and (7.2) we have

$$\frac{d}{dt} f(t,\xi) = Q_+(f)(\xi) - Q_-(f)(\xi) \leq \widehat{J}\,\widehat{I} - 2\pi c_0 f(t,\xi)\,. \tag{7.9}$$

This differential inequality implies

$$f(t,\xi) \leq \max\left\{ \frac{\widehat{J}\,\widehat{I}}{2\pi c_0}\,,\ \|f_0\|_{\mathbf{L}^\infty} \right\}$$

for evry $t \geq 0$ and a.e. $\xi \in I\!R^3$. □

Appendix: O.D.E. theory in Banach spaces

We review here some existence and uniqueness results for abstract O.D.E's, which were used in Section 6.

Let E be a Banach space with norm $\|\cdot\|$, and let $g : E \mapsto E$ be a Lipschitz continuous mapping, so that

$$\big\|g(u) - g(v)\big\| \leq L\,\|u - v\| \qquad\qquad u, v \in E\,. \tag{A.1}$$

It is well known that in this case the Cauchy problem

$$\dot u = g(u)\,, \qquad\qquad u(0) = u_0 \tag{A.2}$$

has a unique solution, which can be obtained as the fixed point of the Picard integral operator $u \mapsto \mathcal{P}u$, with

$$\mathcal{P}u(t) = u_0 + \int_0^t g\big(u(s)\big)\,ds\,. \tag{A.3}$$

Moreover, solutions depend continuously on the initial data. Indeed

$$\frac{d}{dt}\big\|u(t) - v(t)\big\| \leq \Big\|g\big(u(t)\big) - g\big(v(t)\big)\Big\| \leq L \cdot \big\|u(t) - v(t)\big\|\,. \tag{A.4}$$

Hence, by Gronwall's lemma,

$$\big\|u(t) - v(t)\big\| \leq e^{Lt}\|u_0 - v_0\| \qquad\qquad t \geq 0\,. \tag{A.5}$$

In order to achieve (A.5), the assumption of Lipschitz continuity can be greatly relaxed. For example, if E is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, assume that g is a continuous mapping such that

$$\big\langle u - v\,,\ g(u) - g(v)\big\rangle \leq L\,\|u - v\|^2 \qquad\qquad u, v \in E\,. \tag{A.6}$$

Then the computations at (A.4) can be replaced by

$$\frac{d}{dt}\big\|u(t) - v(t)\big\|^2 \leq 2\,\Big\langle u(t) - v(t)\,,\ g\big(u(t)\big) - g\big(v(t)\big)\Big\rangle \leq 2L \cdot \big\|u(t) - v(t)\big\|^2\,. \tag{A.7}$$

Hence, for $t \geq 0$,

$$\big\|u(t) - v(t)\big\|^2 \leq e^{2Lt}\|u_0 - v_0\|^2$$

and (A.5) still holds.

We now examine how the one-sided Lipschitz estimate (A.6) can be reformulated in the context of a general Banach space. Observe that, still in a Hilbert space,

$$\frac{\langle u, w\rangle}{\|u\|} = \lim_{s\to 0} \frac{\langle u + sw\,,\ u + sw\rangle^{1/2} - \langle u, u\rangle^{1/2}}{s} = \lim_{s\to 0} \frac{\|u + sw\| - \|u\|}{s}\,. \tag{A.8}$$

Notice that the right hand side of (A.8) is well defined also in an arbitrary Banach space. Indeed, for every fixed $u, w \in E$, the map $s \mapsto \|u + sw\|$ is convex, being the composition of a linear function with a convex one. Moreover, it is globally Lipschitz continuous because

$$\|u - sw\| - \|u - s'w\| \leq |s - s'\| \cdot \|w\|\,.$$

Therefore, there exists the (possibly distinct) limits

$$[u,w]_- \doteq \lim_{s\to 0-} \frac{\|u+sw\|-\|u\|}{s} \leq \lim_{s\to 0+} \frac{\|u+sw\|-\|u\|}{s} \doteq [u,w]_+ \,. \tag{A.9}$$

Motivated by (A.8), for $u, w \in E$ we thus define the **upper and lower semi-inner products**

$$\langle u,w\rangle_+ \doteq \|u\|\cdot[u,w]_+ \,, \qquad\qquad \langle u,w\rangle_- \doteq \|u\|\cdot[u,w]_- \,.$$

In a Hilbert space the norm is smooth and we clearly have $\langle u,w\rangle_+ = \langle u,w\rangle_- = \langle u,w\rangle$. In a general Banach space there holds

$$-\|u\|\,\|w\| \leq \langle u,w\rangle_- \leq \langle u,w\rangle_+ \leq \|u\|\,\|w\| \,.$$

We are now ready to formulate a general result on the well-posedness of the Cauchy problem for an O.D.E. in a Banach space. In the following, the distance of a point v from the set Ω is of course defined as

$$\text{dist.}\,(v;\Omega) \doteq \inf_{\omega\in\Omega} \|v-\omega\| \,.$$

Theorem A1 (Global existence and uniqueness of solutions). *Consider:*

- *A Banach space E, with norm $\|\cdot\|$ and lower semi-inner product $\langle\cdot,\cdot\rangle$.*
- *A closed, convex subset $\Omega \subset E$.*
- *A bounded, continuous mapping $g : \Omega \mapsto E$ satisfying the one-sided Lipschitz condition*

$$\big\langle u-v\,,\ g(u)-g(v)\big\rangle_- \ \leq\ L\,\|u-v\|^2 \qquad\qquad u,v\in\Omega \tag{A.10}$$

and the tangency condition

$$\liminf_{h\to 0+} \frac{1}{h}\,\text{dist.}\,\big(u+hg(u)\,;\ \Omega\big) = 0 \qquad\qquad u\in\Omega\,. \tag{A.11}$$

In the above setting, for every initial data $u_0 \in \Omega$, the Cauchy problem

$$\frac{d}{dt}u = g(u)\,, \qquad\qquad u(0) = u_0 \tag{A.12}$$

has a unique solution $t \mapsto u(t) \in \Omega$, defined for all $t \geq 0$.

Proof. Fix an arbitrarily large time interval $[0,T]$. Following [M], the solution will be constructed as a limit of polygonal approximations.

1. By the boundedness assumption, there exists a constant M such that

$$\big\|g(u)\big\| \leq M \qquad\qquad u\in\Omega\,. \tag{A.13}$$

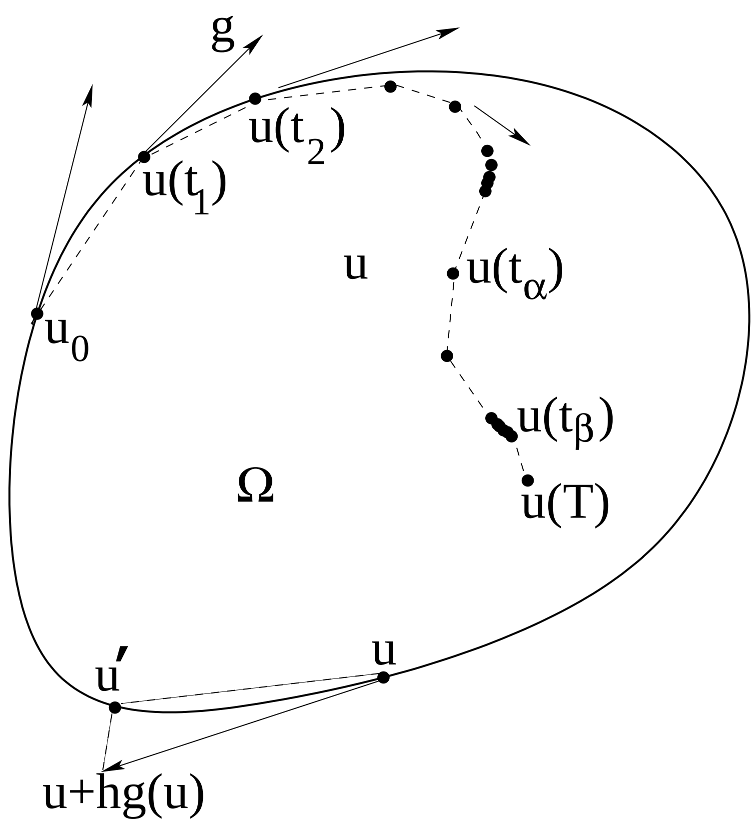

figure 11

Let $\varepsilon > 0$ be given. For any $u \in \Omega$, by the continuity of g and the tangency condition (A.11), we can find $u' \in \Omega$ and $h > 0$ such that

$$\frac{\big\|u' - u - h\,g(u)\big\|}{h} \leq \frac{\varepsilon}{2}\,, \qquad \big\|g(u) - g(v)\big\| \leq \frac{\varepsilon}{2} \qquad \text{if} \ \ \|u - v\| \leq (M+1)h\,.$$

As a consequence, the linear map

$$s \mapsto v(s) = u + s(u' - u) \qquad\qquad s \in [0, h]$$

satisfies

$$u(s) \in \Omega\,, \qquad \Big\|\dot u(s) - g\big(u(s)\big)\Big\| \leq \varepsilon \qquad (A.14)$$

for $s \in [0, h]$.

2. Given $\varepsilon > 0$, we construct an ε-approximate solution $u : [0, \tau] \mapsto \Omega$ by transfinite induction (fig. 11). Namely, by Step 1 there exists $t_1 > 0$ and affine map $u : [0, t_1] \mapsto \Omega$ such that (A.14) holds for $t \in [0, t_1]$.

Next, assume that a piecewise affine map $u : [0, \tau[\, \mapsto \Omega$ has been constructed, and satisfies (A.14) for a.e. $t < \tau$. Here $\tau = \sup_{\alpha \prec \beta} t_\alpha$. Since g is bounded, the map u is Lipschitz continuous, hence the value $u(\tau) = \lim_{t \to \tau-} u(t)$ is certainly well defined. If $\tau = T$ we are done. Otherwise, we have two cases:

(i) If β is a limit ordinal, we simply define

$$t_\beta = \sup_{\alpha \prec \beta} t_\alpha\,, \qquad\qquad u(t_\beta) = \lim_{t \to t_\beta -} u(t)\,.$$

(ii) If β is a successor ordinal, then we apply step 1 with $u = u(t_{\beta-1})$.

This allows us to extend the ε-approximate solution to a larger interval $[0, t_\beta]$. Since $t_{\alpha+1} > t_\alpha$ for every α, we must have $t_{\alpha^*} \geq T$ for some countable ordinal α^*.

3. Finally, consider a sequence of approximate solutions u^ε, with $\varepsilon \to 0$. We claim that this sequence is Cauchy. Indeed, let u^ε, v^ε be two ε-approximate solutions. By our previous construction we can find a decomposition of the form

$$[0,T] = \left(\bigcup_\alpha I_\alpha\right) \cup \mathcal{N}\,,$$

where the I_α are countably many open intervals where u^ε and v^ε are both affine, and $\mathcal{N}$ has measure zero. The distance function $t \mapsto \big\|u^\varepsilon(t) - v^\varepsilon(t)\big\|$ is Lipschitz continuous. Moreover, recalling (A.9), the dissipativity assumption (A.10) implies

$$\begin{aligned}\frac{d}{dt}\big\|u^\varepsilon(t) - v^\varepsilon(t)\big\| &= \Big[u^\varepsilon(t) - v^\varepsilon(t)\,,\ \dot u^\varepsilon(t) - \dot v^\varepsilon(t)\Big]_- \\ &\leq \Big[u^\varepsilon(t) - v^\varepsilon(t)\,,\ g\big(u^\varepsilon(t)\big) - g\big(v^\varepsilon(t)\big)\Big]_- + 2\varepsilon \\ &\leq L\,\big\|u^\varepsilon(t) - v^\varepsilon(t)\big\| + 2\varepsilon\,.\end{aligned}$$

for a.e. $t \in [0,T]$. In turn, this implies

$$\big\|u^\varepsilon(t) - v^\varepsilon(t)\big\| \leq 2\varepsilon\,\frac{e^{Lt}}{L}\,.$$

As $\varepsilon \to 0$, we thus have the convergence $u^\varepsilon \to u$ uniformly on $[0,T]$. The function $u(\cdot)$ is clearly a solution to the Cauchy problem, because of the continuity of g. □

Remark. As usual, the boundedness assumption on g can be replaced by the assumption of sub-linear growth

$$\big\|g(u)\big\| \leq C\left(1 + \|u\|\right).$$

Basic References

[C] C. Cercignani, *The Boltzmann Equation and its Applications*, Springer, New York, 1988.

[CIP] C. Cercignani, R. Illner and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer-Verlag, 1994.

[M] R. Martin Jr., *Functional Analysis and Differential Equations in Banach Spaces*, Wiley Interscience, 1975.

[V] C. Villani, A review of mathematical topics in collisional kinetic theory, in *Handbook of Mathematical Fluid Mechanics*, S. Friedlander and D. Serre Eds., Elsevier Science, 2002.