1 Bifurcations for nonlinear systems

A system of two ODEs depending on a parameter $\mu$ has the form

\[
\begin{align*}
\dot{x} &= f(x, y, \mu), \\
\dot{y} &= g(x, y, \mu).
\end{align*}
\]

We say that a bifurcation occurs if the phase portrait of the planar system changes, as $\mu$ crosses a critical value $\mu^*$. Namely

(i) new equilibrium points appear (or disappear), or else

(ii) a periodic orbit appears (or disappears).

2 Saddle-node and pitchfork bifurcations

These are bifurcations where the number of equilibrium points changes.

The values $(x^*, y^*, \mu^*)$ where these bifurcations can occur are found by solving the system of equations

\[
\begin{align*}
f(x, y, \mu) &= 0, \\
g(x, y, \mu) &= 0, \\
\frac{\partial f}{\partial x}(x, y, \mu)\frac{\partial g}{\partial y}(x, y, \mu) - \frac{\partial f}{\partial y}(x, y, \mu)\frac{\partial g}{\partial x}(x, y, \mu) &= 0.
\end{align*}
\]

The first two equations say that $(x^*, y^*)$ is an equilibrium point. The third equation means that at this point the Jacobian matrix

\[
A = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix}
\]

has zero determinant, hence it is not invertible.
2.1 Saddle-node bifurcations.

As $\mu$ crosses a critical value $\mu^*$ a saddle and a (stable or unstable) node join together and disappear. The standard example is shown in Fig. 1:

$$\begin{cases}
\dot{x} = \mu - x^2, \\
\dot{y} = -y.
\end{cases}$$

For $\mu > 0$ there are two equilibrium points: a saddle at $(-\sqrt{\mu}, 0)$ and a stable node at $(\sqrt{\mu}, 0)$. For $\mu < 0$ there is no equilibrium point.

![Figure 1: A saddle-node bifurcation.](image)

2.2 Pitchfork bifurcations.

These typically occur at the origin and for odd systems, such that

$$f(-x, -y) = -f(x, y), \quad g(-x, -y) = -g(x, y).$$

**Supercritical pitchfork:** As $\mu$ crosses a critical value $\mu^*$ a saddle and two stable nodes join together, leaving a single stable node.

The standard example is shown in Fig. 2:

$$\begin{cases}
\dot{x} = \mu x - x^3, \\
\dot{y} = -y.
\end{cases}$$

For $\mu < 0$ there is only one equilibrium point, namely a stable node at $(0, 0)$.

For $\mu > 0$ there are three equilibrium points: a stable node at $(-\sqrt{\mu}, 0)$, a saddle at $(0, 0)$, and another stable node at $(\sqrt{\mu}, 0)$. 
Figure 2: A supercritical pitchfork bifurcation.

**Subcritical pitchfork:** As $\mu$ crosses a critical value $\mu^*$ a saddle and two unstable nodes join together, leaving a single unstable node.

The standard example is is shown in Fig. 3:

\[
\begin{align*}
\dot{x} &= \mu x + x^3, \\
\dot{y} &= -y.
\end{align*}
\]

For $\mu < 0$ there are three equilibrium points: a saddle at $(-\sqrt{\mu}, 0)$, a stable node at $(0, 0)$, and another saddle at $(\sqrt{\mu}, 0)$. For $\mu > 0$ there is only one equilibrium point, namely a saddle at $(0, 0)$.

Figure 3: A subcritical pitchfork bifurcation.
3 Hopf bifurcations

These are bifurcations where a periodic orbit appears (or disappears).

The Hopf bifurcation occurs at an equilibrium point where

(i) The eigenvalues of the Jacobian matrix $A$ in (3) are purely imaginary,

(ii) As $\mu$ crosses a critical value $\mu^*$, the real part of the eigenvalues changes sign. The equilibrium point thus changes from an unstable spiral to stable spiral.

The values $(x^*, y^*, \mu^*)$ where a Hopf bifurcation can occur are found by solving the system of equations

\[
\begin{align*}
    f(x, y, \mu) & = 0, \\
    g(x, y, \mu) & = 0, \\
    \frac{\partial f}{\partial x}(x, y, \mu) + \frac{\partial g}{\partial y}(x, y, \mu) & = 0.
\end{align*}
\]

The first two equations say that $(x^*, y^*)$ is an equilibrium point. The third equation implies that at this point the Jacobian matrix $A$ has zero trace.

3.1 Supercritical Hopf bifurcation.

As $\mu$ crosses a critical value $\mu^*$, a stable periodic orbit and an unstable equilibrium point join together, leaving a stable equilibrium point.

The standard example is shown in Fig. 4:

\[
\begin{align*}
    \dot{x} & = \mu x + y - x^3, \\
    \dot{y} & = -x + \mu y - y^3.
\end{align*}
\]

For $\mu < 0$ the origin is a stable spiral point, while for $\mu > 0$ the origin is an unstable spiral. When $\mu = 0$, the origin is stable. Indeed, in polar coordinates we find

\[
r\dot{r} = x\dot{x} + y\dot{y} = -(x^4 + y^4) < 0.
\]

For $\mu > 0$ small, a topological argument yields the existence of a stable periodic orbit.
Figure 4: A supercritical Hopf bifurcation. At the critical value \( \mu^* = 0 \) the origin is stable. For \( \mu > 0 \) the origin is an unstable spiral point and the phase portrait contains a stable periodic orbit.

3.2 Subcritical Hopf bifurcation.

As \( \mu \) crosses a critical value \( \mu^* \), an unstable periodic orbit and a stable equilibrium point join together, leaving an unstable equilibrium point.

The standard example is shown in Fig. 5:

\[
\begin{align*}
\dot{x} &= \mu x + y + x^3, \\
\dot{y} &= -x + \mu y + y^3.
\end{align*}
\]

For \( \mu < 0 \) the origin is a stable spiral point, while for \( \mu > 0 \) the origin is an unstable spiral. When \( \mu = 0 \), the origin is unstable. Indeed, in polar coordinates we find

\[
r\dot{r} = x\dot{x} + y\dot{y} = x^4 + y^4 > 0.
\]

For \( \mu < 0 \) small, a topological argument yields the existence of an unstable periodic orbit.

Figure 5: A subcritical Hopf bifurcation. At the critical value \( \mu^* = 0 \) the origin is unstable. For \( \mu < 0 \) the origin is a stable spiral point and the phase portrait contains an unstable periodic orbit.
Note: To distinguish between supercritical and subcritical Hopf bifurcation, the key is to understand the stability of the equilibrium when $\mu = \mu^*$. In this case, the linearization always indicates that the equilibrium is a center, providing no useful information. To understand whether it is stable or not, we must look at higher order terms. Usually, this can be done writing the system in polar coordinates. Consider the ODE

$$
\frac{dr}{d\theta} = \frac{rx\dot{x} + y\dot{y}}{xy - y\dot{x}} = \frac{r}{x f(x,y) + y g(x,y)} - \frac{r}{f(x,y) - y f(x,y)}.
$$

with initial data

$$r(0) = \varepsilon,$$

with $\varepsilon > 0$ small. Check if $r(2\pi)$ is larger or smaller than $r(0)$. Depending on the direction of rotation, this yields the desired information about stability (Fig. 6).

Figure 6: Deciding the stability of the origin, using polar coordinates. Assume that, when $\varepsilon > 0$ is small enough, the solution of (4) satisfies $r(2\pi) > r(0) = \varepsilon$. Left: if trajectories rotate counterclockwise, then the origin is unstable. Right: if trajectories rotate clockwise, then the origin is stable.