1. Consider the following ODEs for $X(t)$ and $Y(t)$:

\[
\frac{dX}{dt} = -X + 18Y \quad \frac{dY}{dt} = -4Y - X
\]

Write as a single matrix equation for a column vector $(XY)$.

2. Consider the following three reactions:

\[
A + B \xrightarrow{k_1} C \quad \text{(1)}
\]
\[
2A + C \xrightarrow{k_2} D + B \quad \text{(2)}
\]
\[
3A + D \xrightarrow{k_3} E \quad \text{(3)}
\]

(a) Write the ODEs for the concentrations of species in these reactions.

(b) Choose one species to be in quasi-steady equilibrium, and use this to simplify the model.

3. Imagine a swinging pendulum described by $\theta = \theta(t)$, where the rod of length $L_0$ can oscillate like a spring. This means that the mass on the end of the pendulum has two kinds of oscillating motion: one swinging, and one spring-like. Treat these two separately:

(a) Write the force balance acting orthogonal to $L$, which drives the swinging of $\theta$. Note that if $L(t) = L_0$ this should lead to the usual pendulum equation.

(b) Write the force balance acting along $L$. Feel free to assume whatever you need to about the way the length varies.

(c) What are the steady state (“equilibrium”) solutions of these two equations?

(d) If $\theta(0) = 0$, and you pull down on the spring, it will oscillate. Will it also begin to swing?

4. Consider the ODE for $u(t)$:

\[
u'' + \lambda u = 0,
\]

where $\lambda$ and $m \in \mathbb{R}$, and both are positive.

(a) Using the Energy Method, show that the energy associated with this system is constant.

(b) If $m$ varied with time, how would this change the ODE? What would the new ODE be?

(c) Discuss how this would change the energy argument in (a).
5. Consider the ODE for \( u(t) \)

\[
  u'' + u = f
\]

When \( f = 0 \), you should already know that \( u(t) = C_1 \cos t + C_2 \sin t \). Now let \( f = -\epsilon u^3 \).

(a) Write the general form for an asymptotic expansion for \( u(t) \) in \( \epsilon \).

(b) Show that the \( O(1) \) term of this expansion \( u_0 \) is the solution to the ODE when \( f = 0 \).

(c) Derive the equation that the \( O(\epsilon) \) term \( u_1 \) satisfies (you do not need to solve it).

6. Consider the following integral

\[
  I(x) = \int_0^{+\infty} e^{-x \sin \theta} d\theta
\]

Evaluate \( I(x) \) for large \( x \) by using Laplace’s method, which says that if \( x \) is large then the main contribution to the integral will be when \( x \) is multiplied by something close to zero. HINT: Taylor expand to \( O(\theta) \) and integrate.

7. Often a lightbulb will flash and burn out when you first enter the room and turn on the light switch. Why is that? To begin to answer that question, first define the state of the system, and make a list of variables. Which would be the parameters, and which variables would vary dynamically (the observables)?