Solutions to homework 7

7.4-7.5/2
Suppose
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1. \]
Then for large \( n \) there exists \( M, 1 < M < L \), so that
\[ \left| \frac{a_{n+1}}{a_n} \right| > M, \]
for \( n \geq N \). Therefore
\[ |a_{n+1}| > M|a_n|, \quad (1) \]
We can choose a (small) constant \( c \) so that
\[ |a_N| > cM^N. \]
Applying property (1) inductively, we have
\[ |a_n| > cM^n, M > 1, \]
for \( n \geq N \). Therefore \( a_n \) does not converge to zero. By the \( n \)-th term test \( \sum a_n \) does not converge.

7.4-7.5/3
a) Suppose
\[ \lim_{n \to \infty} |a_n|^{1/n} = L > 1. \]
Then for large \( n \) there exists \( M, 1 < M < L \), so that
\[ |a_n| > M^n, \]
for \( n \geq N \). Therefore \( a_n \) does not converge to zero. By the \( n \)-th term test \( \sum a_n \) does not converge. b)
\[ \sum \frac{1}{n} \sum \frac{(-1)^n}{n} \]

7.4-7.5/4
Suppose
\[ \lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = 1. \]
This means that for large \( n \geq N \)

\[
\frac{1}{2} |b_n| < |a_n| < 2|b_n|.
\] (2)

If \( \sum |b_n| \) converges then \( \sum |a_n| \) converges by the comparison theorem with the second inequality in (2). If \( \sum |b_n| \) diverges then \( \sum |a_n| \) diverges by the comparison theorem with the first inequality in (2).

We have

\[
0 \leq b_n = \int_{n}^{n+1} f(x) dx \leq a_n = f(n).
\]

We assume that \( \sum b_n = \int_{1}^{\infty} f(x) dx \) diverges. By the comparison theorem \( \sum a_n = \sum f(n) \) diverges.