Solutions to homework 5

5.1.4
Let us first prove that if \( c_n \to L \), then \( |c_n| \to |L| \) (cf. exercise 5.3.5). We need to investigate three cases \( L > 0 \), \( L < 0 \), \( L = 0 \).

\( L = 0 \) is trivial: given \( \varepsilon > 0 \)
\[
|c_n - 0| < \varepsilon \iff |c_n - 0| < \varepsilon,
\]
for sufficiently large \( n \).

\( L < 0 \). Since for \( c_n \to L \), then for sufficiently large \( n \), \( c_n < 0 \) and
\[
|L - c_n| < \varepsilon \iff L - \varepsilon < c_n < L + \varepsilon < 0,
\]
multiplying the last inequalities by \(-1\) we have
\[
0 < |L| - \varepsilon < |c_n| < |L| + \varepsilon \iff |L| - |c_n| < \varepsilon.
\]

Similarly, the last case shown:
\[
|L - c_n| < \varepsilon \iff 0 < L - \varepsilon < c_n < L + \varepsilon \iff |L| - |c_n| < \varepsilon.
\]

Let us now turn back to our problem.
By the previous argument, given \( \varepsilon > 0 \)
\[
|L| - \varepsilon < \frac{|a_n|}{|b_n|} < |L| + \varepsilon \iff (|L| - \varepsilon)|b_n| < |a_n| < (|L| + \varepsilon)|b_n|.
\]
Since \( |b_n| \to 0 \), by the \( K - \varepsilon \) principle and the Squeeze Theorem \( |a_n| \to 0 \).

5.2.3.
Imitating Example 5.2C we have
\[
\int_{n-1}^{n} \sqrt{x} \, dx < \sqrt{n} < \int_{n}^{n+1} \sqrt{x} \, dx,
\]
hence
\[
\int_{0}^{n-1} \sqrt{x} \, dx = \frac{2}{3} (n - 1)^{3/2} < \sum_{k=1}^{n} \sqrt{k} < \int_{1}^{n} \sqrt{x} \, dx = \frac{2}{3} n^{3/2} - \frac{2}{3}.
\]
We have
\[
\lim_{n \to \infty} \frac{\frac{2}{3} n^{3/2} - \frac{2}{3}}{\frac{2}{3} n^{3/2}} = 1,
\]
\[
\lim_{n \to \infty} \frac{2(n - 1)^{3/2}}{3n^{3/2}} = 1.
\]

Hence by the Squeeze Theorem
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt[k]{i} = 1,
\]
which, by definition, means
\[
\sum_{i=1}^{n} \sqrt[k]{i} \sim \frac{2}{3} n^{3/2}.
\]

5.4.2.

If \( n_i \) is a subsequence of prime numbers then \( s(n_i) = n_i \), and
\[
\lim_{n_i \to \infty} \frac{s(n_i)}{n_i} = 1.
\]

If \( n_j = 2^k \), then \( s(n_j) = 2k = 2 \ln n_j \), and
\[
\lim_{n_j \to \infty} \frac{s(n_i)}{n_i} = 2 \lim_{k \to \infty} k(0.5)^k = 0,
\]
by exercise 3.4.4.

Hence the sequence \( s(n)/n \) has two subsequences, that converge to different limits. By the subsequence theorem, \( s(n)/n \) does not converge.