Solutions to homework 11

12.1.4
a) Observe that the maximum of \( f(x) = xe^{-x} \) is located at \( x = 1 \). Indeed, by Calculus
\[
(xe^{-x})' = (1 - x)e^{-x} = 0 \iff x = 1.
\]
At \( x = 0 \), \( f(x) = 0 \). At \( x = 1 \), \( f(x) = 1/e \). Hence by the intermediate value theorem there is \( c \in [0, 1] \) such that
\[
f(c) = a,
\]
for any \( a \in [0, 1/e] \).
b) For the second part observe that
\[
\lim_{x \to +\infty} xe^{-x} = 0.
\]
Hence for any \( a > 0 \), there is \( x_0 \) such that
\[
f(x_0) < a.
\]
Again, by the intermediate value theorem there is \( d \in [1/e, x_0] \) such that
\[
f(d) = a,
\]
for any \( a \in (0, 1/e) \).
By inspection, \( d \neq 1/e \), hence we found two distinct points \( c \) (in part a)) and \( d \) (in this part) so that
\[
x^{-x} = a \text{, when } x = c, \text{ and } x = d.
\]
12.2.3
Observe that for \( x_n' = \pi n \)
\[
f(x_n') = \pi n - \tan \pi n = \pi n > 0.
\]
Since for every fixed \( n \)
\[
\lim_{x \to (\pi/2 + \pi n)^-} \tan x = +\infty,
\]
there exists \( x_n'' < \pi/2 + \pi n \) such that
\[
\tan x_n'' > \pi/2 + \pi n > x_n''.
\]
Hence
\[ f(x_n'') = x_n''' - \tan x_n'' < 0. \]

Therefore \( f(x) \) changes sign on every interval \([x_n', x_n'']\). Since \( f(x) \) is defined and continuous on each \([x_n', x_n'']\), by the intermediate value theorem there is a zero on each \([x_n', x_n'']\). Since there are infinitely many such (nonintersecting) intervals, there are infinitely many zeroes of \( f(x) \).

13.4.1
Suppose \( I = [a, b] \) and \( f(I) = [c, d] \).

a) By definition of the supremum and the infimum
\[
c = \inf_{x \in [a,b]} f(x), \quad d = \sup_{x \in [a,b]} f(x). \]

By definition of the image, there are \( x_1, x_2 \in [a, b] \) so that
\[
f(x_1) = d, \quad f(x_2) = c. \]

Hence the Maximum theorem holds.

b) Let us assume for definiteness that \( f(a) < f(b) \). Consider any \( k \in [f(a), f(b)] \). Since \( f(a) \in [c, d] \) and \( f(b) \in [c, d] \), we have that \( k \in [c, d] \). By definition of the image, there is \( x_3 \in [a, b] \) so that
\[
k = f(x_3). \]

Hence the intermediate value property holds.

13.5.6

a) By boundedness of the secants assumption for any \( x_1 \) and \( x_2 \)
\[
|f(x_1) - f(x_2)| \leq K|x_1 - x_2|. \]

Given \( \varepsilon > 0 \), choose \( \delta = \varepsilon/K \) and we have
\[
|f(x_1) - f(x_2)| \leq K|x_1 - x_2| < \varepsilon, \]
for \( |x_1 - x_2| < \delta \).

b) No, \( \sqrt{x} \) does not satisfy the hypothesis at \( x_1 = 0 \): for \( x_2 \approx 0 \) slopes can be arbitrarily large, because
\[
\lim_{x \to 0} \frac{\sqrt{x}}{x} = +\infty. \]

The function \( f(x) \) is, however, uniformly continuous, because it is continuous on the compact interval \([0, 1]\) and therefore we can apply the uniform continuity theorem.