Solutions to homework 10

11.3.1
The comparison of areas gives
\[
\frac{\cos x \sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}.
\]
Since \( \sin x > 0 \), dividing the above inequality by \( 2 \sin x \) we have
\[
\cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.
\]
By the reciprocal law
\[
\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}.
\]
Since \( \cos x \) is continuous at \( x = 0 \), \( \lim_{x \to 0^+} \cos x = 1 \). By the quotient theorem
\[
\lim_{x \to 0^+} \frac{1}{\cos x} = 1.
\]
By the Squeeze theorem
\[
\lim_{x \to 0^+} \frac{\sin x}{x} = 1.
\]
Since \( \sin x \) is odd for every \( x < 0 \) we have
\[
\frac{\sin x}{x} = \frac{\sin(-|x|)}{-|x|} = -\frac{\sin |x|}{-|x|} = \frac{\sin |x|}{|x|}.
\]
Since \( |x| > 0 \) we can use the previous result
\[
\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{|x| \to 0} \frac{\sin |x|}{|x|} = 1.
\]
Hence
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
11.4.1
\[
|f(x) - f(0)| = |\sqrt{x} \cos \frac{1}{x}| \leq \sqrt{x}
\]
Inequality \( \sqrt{x} < \varepsilon \) holds when \( |x| < \varepsilon^2 \). Hence if we take \( \delta = \varepsilon^2 \), we have
\[
|f(x) - f(0)| < \sqrt{\delta} = \varepsilon,
\]
for all $x \in [0, \delta)$.

11.4.3
The discontinuities of $h(x) = f(g(x))$ are located at all the zeroes of $g(x)$, because

$$h(x) = \begin{cases} 
1, & \text{if } f(x) > 0, \\
0, & \text{if } f(x) = 0, \\
-1, & \text{if } f(x) < 0,
\end{cases}$$

They are jump discontinuities.

11.4.4
We will prove the case for $h = \max(f, g)$ only. The proof for the minimum is identical. We denote by $D$ the set where $f$ and $g$ are continuous. Consider any point $x_0 \in D$. There are two cases: either $g(x_0) \neq f(x_0)$ or $g(x_0) = f(x_0)$. Consider the first case. Suppose for definiteness that at $f(x_0) > g(x_0)$. Then by the positivity theorem 11.4B there exists a $\delta$-neighborhood of $x_0$ where $f(x) > g(x)$ for all $x$ in this neighborhood: $x \in D, |x - x_0| < \delta$. Then for any $x \in D, |x - x_0| < \delta h(x) = f(x)$. Therefore $h(x)$ is continuous at $x_0$, because $f(x)$ is continuous at $x_0$.

Consider now the point $x_0$ where $g(x_0) = f(x_0)$. By definition of $h$

$$|h(x) - h(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)|.$$ 

Since $f$ and $g$ are continuous at $x_0$, for any $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - f(x_0)| < \frac{\varepsilon}{2}, \quad |g(x) - g(x_0)| < \frac{\varepsilon}{2},$$

for any $x \in D$ and $|x - x_0| < \delta$. Therefore

$$|h(x) - h(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

for any $x \in D$ and $|x - x_0| < \delta$. Hence $h(x)$ is continuous at such $x_0$.