Math 312, Fall 2004
Final
Total 100 pts
“Prove” means give a careful, well-explained proof.
Put your name on the exam.
Good luck!

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
</tr>
</tbody>
</table>
1. Give the definition of a uniformly continuous function on $[a, b]$. 
2. State the Cauchy criterion for convergence of sequences and prove it.
Reminder: the Cauchy criterion is about convergence of sequences \( \{a_n\} \) that satisfy:
given \( \varepsilon > 0 \) there is \( N \) so that \( |a_n - a_m| < \varepsilon \) for \( m,n > N \).
3. Define a sequence recursively by $a_{n+1} = \sqrt{2a_n}$, $a_0 > 0$. Prove that the sequence \( \{a_n\} \) is monotone and bounded. Prove that there is the unique limit $L$, independent of $a_0$. 
4. Suppose a function \( f(x) \) in \([a, b]\) is such that for any \([c, d]\), \(a \leq c < d \leq b\)

\[
\sup_{x \in [c,d]} f(x) - \inf_{x \in [c,d]} f(x) \leq \sqrt{d-c}.
\]

Prove that \( f(x) \) has a maximum and minimum on \([a, b]\).

Hint. Prove that \( f(x) \) is continuous on \([a, b]\).
5. Find the radius of convergence $R$ and determine convergence at $x = R, x = -R$ for

$$\sum \left( \frac{n}{n+2} \right)^{n^2} x^n.$$
6. For a function

\[ f(x) = \begin{cases} 
1/q, & x = p/q \in \mathbb{Q}, \ (p, q) = 1, \\
0, & x \not\in \mathbb{Q} 
\end{cases} \]

determine directly from the definitions where \( f(x) \) is
a) continuous,
b) integrable.
7. Suppose a function $f(x)$ is continuous on a closed interval $I = [a, b]$, and that $f(I) = [f(a), f(b)]$. Suppose further that as $x$ varies over $I$, $f(x)$ never repeats a value. Prove $f(x)$ is strictly increasing.
8. Suppose \( f(x) \) is differentiable on the \textit{open} interval \( I = (0, 1) \). Also suppose \( f'(x) \) is (globally) bounded on \( I \). Prove that \( f(x) \) is uniformly continuous on \( I \).

Hint. One way to prove it is to show that the secants of \( f(x) \) has bounded slopes, using the mean-value theorem.
9. Suppose $f(x)$ is differentiable on $[-a, a]$, $f(-a) = f(a) = 0$, $f(0) = 1$. Prove that there are two points $-a < b < 0$ and $0 < c < a$ such that

$$f'(b) = -f'(c).$$

Hint. Use the mean-value theorem.
10. Directly evaluate \( \int_1^a f(x)dx \) where \( f(x) = x^k \), and \( k \) is a positive integer by using upper sums and

\[
\lim_{|P| \to 0} U_f(P) = \int_1^a f(x)dx.
\]

Use the \( n \)-partition \( 1 < r < r^2 < \cdots < r^{n-1} < a \), where \( r = a^{1/n} \).

Remark: You may use here the l’Hospital’s rule:

\[
\lim_{n \to \infty} \frac{a^{1/n} - 1}{a^{(k+1)/n} - 1} = \frac{1}{k + 1}.
\]