M598B: Tensor summation notation vs matrix multiplication

Some students have difficulties with the summation notation in tensors. In particular, students taking a Mechanical Engineering course that uses the book “Incompressible Flow” by R. L. Panton, (2nd edition, John-Wiley) are especially confused, since the author Panton uses his own notation. Here is a clarification.

In our class notation, we have

\[ i_i' \cdot i_j = \alpha_{i'j}. \]  

(1)

Note that our primed index is always the first index in \( \alpha \) and we do not need or use \( \alpha_{lk'} \). Our coordinate transformation and its inverse are

\[ x'_l = \alpha_{lk} x_k + x_0l, \]

\[ x_l = \alpha_{lk'} x'_k + x'_0l. \]  

(2)

The first equation of (2) is the transformation from \( K \) to \( K' \), while the second equation of (2) is the inverse transformation. Note the index summed in the first equation is the second index of \( \alpha \), while the index summed in the second equation of (2) is the first index.

The \( (\alpha_{lk}) \) are often given as a 3 \( \times \) 3 matrix

\[ \alpha = (\alpha_{lk}) = \begin{pmatrix} \alpha_{1'1} & \alpha_{1'2} & \alpha_{1'3} \\ \alpha_{2'1} & \alpha_{2'2} & \alpha_{2'3} \\ \alpha_{3'1} & \alpha_{3'2} & \alpha_{3'3} \end{pmatrix}. \]

In the transformation of first-order tensors we have

\[ A' = \alpha A \]

where the matrix multiplication of a 3 \( \times \) 3 matrix \( \alpha \) with a column vector \( A \) is used. Or more explicitly:

\[ \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = \begin{pmatrix} \alpha_{1'1} & \alpha_{1'2} & \alpha_{1'3} \\ \alpha_{2'1} & \alpha_{2'2} & \alpha_{2'3} \\ \alpha_{3'1} & \alpha_{3'2} & \alpha_{3'3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \]
For the inverse transformation we have

\[ A_r = A'_r \alpha \]

where \( A_r \) and \( A'_r \) are arranged in the row form. More explicitly we have

\[
(a_1, a_2, a_3) = (a'_1, a'_2, a'_3) \begin{pmatrix}
\alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\
\alpha'_{21} & \alpha'_{22} & \alpha'_{23} \\
\alpha'_{31} & \alpha'_{32} & \alpha'_{33}
\end{pmatrix}.
\]

Again the matrix multiplication of a row vector \( A'_r \) with the matrix \( \alpha \) is used. To use column as the default, we can write the inverse transform as

\[ A = \alpha^T A' \]

where \( \alpha^T \) denotes the transpose of \( \alpha \).

For the transformation law of second-order tensors, we have

\[ A' = \alpha A \alpha^T \]

where \( A' \) and \( A \) are matrices \( (a'_{ij}) \) and \( (a_{ij}) \). One can verify this by checking on the individual component

\[ a'_{ij} = \alpha_{il} a_{lm} \alpha_{jm}. \]

Although we use only \( \alpha_{i'j} \):

\[ \alpha_{i'j} = ( \text{by definition} ) \ i'_l \cdot i_j, \]

some authors, in particular, Panton, use also the notation \( \alpha_{ji'} \), which are defined as

\[ \alpha_{ji'} = ( \text{by definition} ) \ i_j \cdot i'_l = \alpha_{i'j}. \]

(Note that \( \alpha_{ji'} = \alpha_{i'j} \) does not imply that the matrix \( \alpha \) is symmetric.) With the new notation \( \alpha_{ji'} \), we can express the coordinate transformation and its inverse as

\[
x'_l = \alpha_{k'l} x_k + x_{0l},
x_l = \alpha_{k'l'} x'_k + x'_{0l}.
\]

And the second-order tensor transformation law is

\[ a'_{ij} = \alpha_{i'k} a_{lm} \alpha_{mj'}. \]

These look very nice and unified from some point of view.