3.3. Bounded linear operators and adjoint operators.


Similar to the linear transformations $L$ from a Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$ represented by

$$y = Ax,$$

we define a **linear operator** $L$ from a Hilbert space $H$ to $H$ to be a mapping that satisfies

$$L(\alpha f + \beta g) = \alpha Lf + \beta Lg$$

for all real numbers $\alpha$ and $\beta$ and all members $f$ and $g$ in $H$. The linear operator is called **bounded** if there exists a constant $C$ such that

$$\|Lf\| \leq C\|f\|$$

for all $f \in H$.

Let us look at an example. From the differential equation

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1$$

with the two-point boundary value

$$u(0) = u(1) = 0,$$

one can obtain the solution formula

$$u(x) = \int_0^1 k_0(x, y)f(y)dy$$

where

$$k_0(x, y) = \begin{cases} 
  y(x - 1), & 0 \leq y < x \leq 1 \\
  x(y - 1), & 0 \leq x < y \leq 1.
\end{cases}$$

This solution formula is a bounded linear operator for $f(x) \in L^2[0,1]$ to $u(x) \in L^2[0,1]$, see the next theorem.

**Theorem.** For any $k(x, y)$ such that

$$\int_a^b \int_a^b k^2(x, y)dxdy = C < \infty,$$
the operator

\[ Tu(x) = \int_a^b k(x, y)u(y)dy \]

is a bounded linear operator from \( L^2[0, 1] \) to \( L^2[0, 1] \).

This operator is called a **Hilbert-Schmidt operator**.

**Proof.** We use Cauchy-Schwarz inequality

\[
\|Tu(x)\| = (\int_a^b (Tu(x))^2 dx)^{1/2} \\
= (\int_a^b \int_a^b k(x, y)u(y)dy)^{1/2} \\
\leq (\int_a^b (\int_a^b k^2(x, y)dy)(\int_a^b u^2(y)dy)dx)^{1/2} \\
= (\int_a^b \int_a^b k^2(x, y)dydx)^{1/2}(\int_a^b u^2(y)dy)^{1/2} \\
= C\|u\|. 
\]

The proof is complete.

We introduce an important concept.

**Definition.** The adjoint operator \( T^* \) of an operator \( T \) in a Hilbert space \( H \) is an operator such that

\[
<Tf, g> = <f, T^*g>
\]

for all \( f \) and \( g \) in \( H \).

We know that the adjoint of a real matrix \( A \) is the transpose \( A^t \).

For a Hilbert-Schmidt operator, the adjoint is

\[ T^*u(x) = \int_a^b k(y, x)u(y)dy. \]

The proof is as follows: we have

\[
<Tu, v> = \int_a^b (Tu)(x)v(x)dx = \int_a^b \int_a^b k(x, y)u(y)v(x)dydx.
\]

We then change the order of integration to obtain

\[
<Tu, v> = <u, T^*v>.
\]

In general, we have the following theorem.

**Theorem.** The adjoint of a bounded linear operator of a Hilbert space always exists and is bounded linear.

Proof is ommitted.
The application of the adjoint operator is given in the following theorem.

**Theorem** (Fredholm Alternative Theorem) Suppose $L$ is a bounded linear operator in a Hilbert space $H$ with closed range. Then the equation

$$Lf = g$$

has a solution if and only if

$$< g, v > = 0$$

for all $v$ such that $L^*v = 0$. The term “closed range” is a technical term which means that the image of $L$: $\{g \mid g = Lf, f \in H\}$ is a closed set in $H$.

The proof of the “only if” part is as follows. Suppose a solution $f$ exists for a $g$, then

$$< g, v > = < Lf, v > = < f, L^*v > = < f, 0 > = 0$$

for all $v$ such that $L^*v = 0$. The “if” part is omitted.

In terms of matrices we see that $Ax = y$ has a solution if and only if $y$ is orthogonal to all the solutions $v$ such that $A^t v = 0$.

### 3.4. Spectral theory for compact operators.

A bounded linear operator is called **compact** if it maps any bounded sequence $\{f_n\}_{n=1}^\infty$ into a sequence $\{Lf_n\}_{n=1}^\infty$ that has a convergent subsequence.

Any linear transformation in $\mathcal{R}^n$ is compact.

**Theorem.** Any Hilbert-Schmidt operator is compact.

For a compact operator, we have a spectral theory just like the eigenvalue problem for matrices. For a square matrix $A$, if there exist a number $\lambda$ and a nonzero vector $u$ such that there holds

$$Au = \lambda u$$

then $\lambda$ is called an **eigenvalue** of $A$ and $u$ is called an associated **eigenvector**. For a large class of $n \times n$ matrices $A$, we know that there exist $n$ eigenvalues and associated eigenvectors.

For any linear operator the definition of eigenvalues and eigenvectors are the same. If there exist a number $\lambda$ and a nonzero member $u$ such that there holds

$$Lu = \lambda u$$
then $\lambda$ is called an **eigenvalue** of $L$ and $u$ is called an associated **eigenfunction**.

For a compact linear operator $L$, we can prove that there exists at least one eigenvalue and an associated eigenfunction. Under general conditions, a compact linear operator $L$ has infinitely many eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots,$$

with associated eigenfunctions

$$u_1, u_2, u_3, u_4, \ldots.$$ These eigenfunctions form an orthogonal basis for $H$ so that any function in $H$ can be written as an infinite series

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots.$$ A bounded linear operator on $H$ can be represented by an infinite matrix.

**Example.** Consider

$$\frac{d^2 u}{dx^2} + \lambda u = g(x), \quad 0 < x < 1$$
with boundary conditions

$$u(0) = u(1) = 0.$$ Any solution $u$ will also satisfy the integral equation

$$u(x) + \lambda \int_0^1 k_0(x, y)u(y)dy = G(x)$$
where

$$G(x) = \int_0^1 k_0(x, y)g(y)dy.$$ When $g(x) = 0$, we have an eigenvalue problem

$$\int_0^1 k_0(x, y)u(y)dy = -\frac{1}{\lambda}u.$$ One can verify that we have

$$\lambda = (n\pi)^2, \quad n = 1, 2, 3, \ldots;$$
and

$$u = \sin(n\pi x).$$
The verification can be done through the differential equation rather than the integral equation.

If \( \lambda \) is not equal to any of the eigenvalues, then the equation

\[
Lu(x) = G(x),
\]

where

\[
Lu(x) = u(x) + \lambda \int_0^1 k_0(x, y)u(y)dy,
\]

has a unique solution for any \( g(x) \). If \( \lambda \) is one of the eigenvalues, then \( g(x) \) needs to be orthogonal to all the solutions \( v \) to

\[
L^*v(x) = v(x) + \lambda \int_0^1 k_0(y, x)v(y)dy = 0.
\]

We note that \( T \) is called **self-adjoint** if \( T^* = T \). So the function \( k_0(x, y) \) yields a self-adjoint operator. A Hilbert-Schmidt operator is self-adjoint when \( k(x, y) = k(y, x) \). The eigenvalues of self-adjoint compact linear operators are all real numbers.

—End of Chapter III—

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