3.1. Banach and Hilbert spaces (Continued)


But we do wonder: If $S$ is complete with respect to one norm, is it complete with respect to other norms?

Let us look at the set of all continuous functions $C$. It has an inner product. The inner product induces a norm

$$
\|f\|_{L^2} = \left( \int_{a}^{b} (f(x))^2 \, dx \right)^{1/2}.
$$

We can prove easily that it is indeed a norm. This is an important norm for applications. We wonder whether the set $C$ is complete under this norm.

Let us consider a sequence of continuous functions:

$$
f_n(t) = \begin{cases} 
0, & 0 \leq t < \frac{1}{2} - \frac{1}{n}, \\
\frac{1}{2} + \frac{n}{2}(t - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n} \\
1, & \frac{1}{2} + \frac{1}{n} < t \leq 1.
\end{cases}
$$

From its graph (see text book) it is clear that it "converges" to a function

$$
f_0(t) = \begin{cases} 
0, & 0 \leq t < \frac{1}{2}, \\
\frac{1}{2}, & t = \frac{1}{2} \\
1, & \frac{1}{2} < t \leq 1.
\end{cases}
$$

We show this convergence is not in the sup-norm:

$$
\|f_n - f_0\| = \max_{t \in [0,1]} |f_n(t) - f_0(t)| = 1,
$$

which does not converge to zero. However, we can show that it converges to $f_0$ in the $L^2$ norm as follows. We first have

$$
\|f_n - f_0\|_{L^2} = \left( \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} (f_n(t) - f_0(t))^2 \, dt \right)^{1/2}
= \left( \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{n}} (f_n(t) - f_0(t))^2 \, dt \right)^{1/2},
$$

(1)

where we have shortened the interval of integration since the difference between $f_n(t)$ and $f_0(t)$ is zero outside the middle interval. Note that the middle portion
shrinks in length to zero while the integrand is no larger than 1, we have

\[
\|f_n - f_0\|_{L^2} \leq \left( \int_\frac{1}{2}^{\frac{1}{2} + \frac{1}{n}} (1)^2 \, dt \right)^{\frac{1}{2}}
\]

\[
= \left( \left( \frac{1}{2} + \frac{1}{n} \right) - \left( \frac{1}{2} - \frac{1}{n} \right) \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{2}{n} \right)^{\frac{1}{2}}
\]

which goes to zero. This shows that \( f_0 \) is a limit in the \( L^2 \) norm. But \( f_0 \) does not belong to \( C \). This is rather like the limit \( \sqrt{2} \) of rational numbers. When we collect all possible functions that are limits of continuous functions on \([a, b]\) in the \( L^2 \) norm, we get a larger class of functions than \( C[a, b] \).

(We showed in class that this sequence is a Cauchy sequence.)

**Definition 3.5.** The collection of all possible functions that are limits of continuous functions on \([a, b]\) in the \( L^2 \) norm is called the space \( L^2[a, b] \).

It turns out that \( L^2[a, b] \) is complete with respect to its norm \( L^2 \). It is a Banach space. By the next definition, it is also a Hilbert space.

**Definition 3.6 (Hilbert space).** A Banach space whose norm is induced by an inner product is called a Hilbert space.

The ordinary Euclidean space \( R^n \) with the square-root norm is a Hilbert space.

The space of all possible functions that are limits of continuous functions on \([a, b]\) in the \( L^p \) norm \((1 \leq p < \infty)\):

\[
\|f\|_{L^p} = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}
\]

is called the space \( L^p[a, b] \). It is complete and a Banach space. Except for \( p = 2 \), all other \( L^p[a, b] \) spaces are not Hilbert spaces.

**Example.** The function

\[
g(x) = \frac{1}{|x|^\frac{3}{4}} \in L^2[-1, 1]
\]

which is not only discontinuous, but also unbounded.

To get to know these \( L^p \) spaces well, one needs to take the course “Real Analysis.”