Chapter III. Applied Functional Analysis

Course contents outline:

3.1. Normed vector (linear) spaces of functions, Cauchy sequence, completeness, Banach and Hilbert spaces;

3.2. Bounded linear functional and operator; Riesz Representation theorem, adjoint operator,

3.3. Fredholm alternative theorem;

3.4. spectral theory for compact operator.

3.1. Banach and Hilbert spaces


Motivation. When we solve algebraic equations such as

\[ x^2 - 2 = 0, \]

we need numbers, really we need not only rational numbers, but all real numbers and even complex numbers. When we solve differential equations we will need functions. So we now study sets of functions.

Consider the set \( C \) of all real continuous functions on a closed interval \([a, b]\). We know that the sum of two such continuous functions is still a continuous function, and a scalar number multiplying a continuous function yields still a continuous function. As a matter of fact, these two operations satisfy all the eight properties in the definition of a vector (linear) space. Thus \( C \) with these operations is a vector space.

Additionally, the operation

\[ <f, g> = \int_{a}^{b} f(x)g(x)dx \]

qualifies as an inner product, where \( f \) and \( g \) are two members of \( C \). We recall the four defining properties of an inner product:

1. \( <f, f> \geq 0 \) for all \( f \), \( <f, f> = 0 \) only when \( f = 0 \);
2. \( <f, g> = <g, f> \);
3. \( <f + g, h> = <f, h> + <g, h> \);
4. \( <\alpha f, g> = \alpha <f, g> \)
for all members of $C$ and all real numbers $\alpha$.

In applications we realize that one important concept is the distance

$$d(f, g)$$

between two members $f$ and $g$. In a vector (linear) space, the distance is the same as the magnitude of the difference:

$$d(f, g) = d(f - g, 0).$$

This is similar to the vector space of the Euclidean space in the plane. Thus the magnitude or norm of a vector is important.

We use the following

$$\|f\|_{C^0} = \max_{x \in [a, b]} |f(x)|$$

as a measure of magnitude for a member $f$ of $C$. Here we use the symbol $\in$ to represent “belong to”. This magnitude will be referred to as the maximum norm, super norm, or uniform norm. To qualify for a norm, as it is widely known in this field, it needs to satisfy three properties

1. $\|f\| > 0$ for all $f \neq 0$, $\|f\| = 0$ when $f = 0$;
2. $\|\alpha f\| = |\alpha| \|f\|$;
3. $\|f + g\| \leq \|f\| + \|g\|$ (triangle inequality)

for all members $f$, $g$, and all real $\alpha$. We can prove that $\|f\|_{C^0}$ satisfies these properties and thus is a norm. So far the set $C$ is a normed vector space.

Other normed spaces that we know are the Euclidean spaces $\mathbb{R}^n$ with norm

$$\|x\| = \left( \sum_{i=1}^{n} (x_i)^2 \right)^{1/2}$$

for $x = (x_1, x_2, \cdots, x_n)$. We note that the same set of vectors can have different norms. For example we can introduce

$$\|x\|_{\text{max}} = \max_{1 \leq i \leq n} |x_i|$$

as a norm for $\mathbb{R}^n$. It can be verified that it is indeed a norm. There are other norms. A metaphor is that I can use the height measurement to select a student from the
class for an award; I can use the weight measurement for selection; or I can use the Final Exam score for selection.

Completeness. Consider a sequence of rational numbers

\[ 2, \ 2.4, \ 2.41, \ 2.414, \ \ldots. \]

This sequence converges to \( \sqrt{2} \), which is not a rational number. It means that there are a lot of “holes” in the rational number system. Any convergent set of real numbers

\[ x_1, x_2, x_3, \ldots, \]

however, converges to a real number. This means that the real number system is complete. We will be specific later about completeness. Completeness is important in approximation theory.

**Definition 3.1 (limit).** A sequence \( \{f_n\}_{n=1}^\infty \) in a normed vector space \( S \) is said to have a limit \( f \) in \( S \) (or to converge to \( f \) in \( S \)), denoted

\[ \lim_{n \to \infty} f_n = f, \]

if there is such an \( f \) in \( S \) and for any \( \epsilon > 0 \), there exists an integer \( N \) (depends on \( \epsilon \)) such that

\[ \|f_n - f\| < \epsilon \]

for all \( n > N \).

A concept of closeness that does not require knowledge of the limiting member is as follows

**Definition 3.2 (Cauchy sequence).** A sequence \( \{f_n\}_{n=1}^\infty \) in a normed vector space \( S \) is called a Cauchy sequence if for any \( \epsilon > 0 \), there exists an integer \( N \) (depends on \( \epsilon \)) such that

\[ \|f_n - f_m\| < \epsilon \]

for all \( n > N, m > N \).

Property: Every convergent sequence is a Cauchy sequence. This can be proved easily, which we do not have time to do though.

Remark: Not every Cauchy sequence converges to a limit in a normed vector space.
Definition 3.3 (Completeness). A normed vector space $S$ is said to be complete if every Cauchy sequence converges to a limit in $S$.

Definition 3.4 (Banach space). A complete normed vector space is called a Banach space.

Theorem. The space $C[a,b]$ of all continuous functions on $[a,b]$ with the sup-norm is a Banach space.

We do not prove the completeness here.

Definition 3.5 (Hilbert space). A Banach space whose norm is induced by an inner product is called a Hilbert space.