1.3. Vector fields

We have mentioned the magnetic field, which is defined as a domain in which a vector of magnetism is defined at every point. Another example is the velocity field in a stream: each water droplet has a velocity. See Figure 1.3.1.

(Figure 1.3.1. Velocity field)

Generally, a vector field is a domain $\Omega$ and a vector function $\mathbf{A}(\mathbf{r})$ defined in it. Furthermore, a vector field may be time-dependent: $\mathbf{A}(\mathbf{r},t)$.

1.3.1. Line integrals and circulation.

We introduce an integral that gives work done by a force field or circulation of velocity around a loop.

Let $\mathbf{A}(\mathbf{r})$ be a vector field with domain $\Omega$. Let $M_1M_2$ be a curve in the domain directed from $M_1$ to $M_2$. Chop the curve into many small pieces, say $n$ pieces. One typical piece is denoted by the end points $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$. See Figure 1.3.2. The work done in this piece is approximately $\mathbf{A}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i$, where $\Delta \mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$, if we imagine that the vector field is a force field. This can also be interpreted as the flow of the vector field in the direction of $\Delta \mathbf{r}_i$. We sum over all such pieces and take the limit as all $\Delta \mathbf{r}_i \to 0$ to define the line integral:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{A}(\mathbf{r}_i) \cdot \Delta \mathbf{r}_i = \int_{M_1M_2} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = \int_{M_1M_2} A_1dx_1 + A_2dx_2 + A_3dx_3. \tag{1}
\]

Here the notation is $\mathbf{A} = A_1\mathbf{i}_1 + A_2\mathbf{i}_2 + A_3\mathbf{i}_3$. 

(Figure 1.3.1. The velocity field of a stream.)
Line integrals give either total work done by the vector field, or total flow of the vector field along the curve $M_1M_2$ in the direction specified.

Total circulation around a contour $L$ is defined as

$$\Gamma = \oint_L \mathbf{A} \cdot d\mathbf{r}.$$  

(Figure 1.3.2. Definition of the line integral.)

**Example 1.3.1a.** Let $\mathbf{A} = (-x_2, x_1, 0)$. Let $L$ be the unit circle: $x_1^2 + x_2^2 = 1, x_3 = 0$ and counter-clockwise. Then

$$\Gamma = \oint_L \mathbf{A} \cdot d\mathbf{r} = \oint_L -x_2dx_1 + x_1dx_2$$

$$= \int_0^{2\pi} -x_2 \cos \theta d\theta + x_1 \cos \theta d\theta$$

$$= \int_0^{2\pi} \sin \theta d\theta + \cos \theta d\theta$$

$$= \int_0^{2\pi} \sin \theta d\theta + \cos \theta d\theta$$

$$= \int_0^{2\pi} \sin \theta d\theta + \cos \theta d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi.$$  

See Figure 1.3.3.

(Figure 1.3.3. Examples of circulation and line integrals.)
Example 1.3.1b. Let \( A = (x_1, x_2, 0) \) and \( L \) as before. Then
\[
\Gamma = \oint_L A \cdot dr = \oint_L x_1 dx_1 + x_2 dx_2
\]
\[
(L : x_1 = \cos \theta, x_2 = \sin \theta, 0 \leq \theta \leq 2\pi)
\]
\[
= \int_0^{2\pi} \cos \theta d\cos \theta + \sin \theta d\sin \theta
\]
\[
= \int_0^{2\pi} \cos \theta \sin \theta d\theta + \sin \theta \cos \theta d\theta
\]
\[
= \int_0^{2\pi} 0 d\theta = 0.
\]

Example 1.3.1c. Let \( A = (x_1, x_2, 0) \) and \( L \) be the line segment: \( 0 \leq x_1 \leq 1, x_2 = 0 \) directed toward the \( x_1 \)-axis. Then
\[
\Gamma = \oint_L A \cdot dr = \int_0^1 x_1 dx_1 = \frac{1}{2} x_1^2 \bigg|_0^1 = \frac{1}{2}.
\]

1.4. Theorems of Gauss, Green, and Stokes.

Recall the Fundamental Theorem of Calculus:
\[
\int_a^b F'(x) \, dx = F(b) - F(a).
\]
Its magic is to reduce the domain of integration by one dimension. We want higher dimensional versions of this theorem.

We want two theorems like
\[
\iint_S \text{(integrand)} \, dS = \int_{\partial S} \text{(another integrand)} \, d\ell
\]
\[
\iiint_V \text{(integrand)} \, dV = \int_{\partial V} \text{(another integrand)} \, dS.
\]
When $S$ is a flat surface, the formula is called Green’s Theorem. When $S$ is curved, it is called Stokes’ Theorem. The volume integral is called Gauss’ Theorem.

**Gauss’ Theorem.** Let $P(x_1, x_2, x_3), Q(x_1, x_2, x_3), R(x_1, x_2, x_3)$ and all their partial derivatives be continuous in a given domain $V$ with boundary $\partial V$. Then

$$
\iiint_V \left( \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3} \right) dV = \iint_{\partial V} (P \cos(n, x_1) + Q \cos(n, x_2) + R \cos(n, x_3)) dS.
$$

Here $n$ is the unit exterior normal to the surface $\partial V$. The term $(n, x_1)$ represents the angle between $n$ and the $x_1$-axis, etc.

Note that the domain $V$ can have holes: $V$ can be a shell (a ball with another concentric, smaller ball removed, in which case the boundary of $V$ consists of two disjoint parts: an exterior surface with normal pointing outside and an interior surface with exterior unit normal pointing to the origin). The boundary $\partial V$ can be allowed to be piecewise smooth. But the functions $P(x_1, x_2, x_3), Q(x_1, x_2, x_3), R(x_1, x_2, x_3)$ and their derivatives are required to be continuous.

The more common form of Gauss’ Theorem is in vector form. Let

$$
\mathbf{A} = (P, Q, R).
$$

Let the *divergence* of the vector $\mathbf{A}$ be

$$
\text{div } \mathbf{A} = \frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3}.
$$

Recall

$$
\mathbf{n} = (n_1, n_2, n_3) = (\cos(n, x_1), \cos(n, x_2), \cos(n, x_3)).
$$

Then Gauss’ Theorem can be written in vector form:

$$
\iiint_V \text{div } \mathbf{A} \, dV = \iint_{\partial V} \mathbf{A} \cdot \mathbf{n} \, dS.
$$

The proof of Gauss’ Theorem is omitted.

**Green’s Theorem.** Given a planar region $S$ bounded by a closed contour $L$. Suppose that $P(x_1, x_2)$ and $Q(x_1, x_2)$ and all their partial derivatives are continuous in the union $S \cup L$. Then

$$
\iint_S \left( \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dS = \oint_L P \, dx_1 + Q \, dx_2,
$$

where $L$ is traversed in the direction such that $S$ appears to the left of an observer moving along $L$.

End of Lecture 3.