Supplemental Materials I. Linear Dependence

Yuxi Zheng

1. Linear Dependence

Definition 1. A set of vectors $x_1, x_2, \cdots, x_n$ is said to be \textit{linearly dependent} if there exists a set of numbers $c_1, c_2, \cdots, c_n$, not all are zero, such that there holds

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = \mathbf{0}.$$  

For example, the vectors

$$\mathbf{A} = (1, 0, 2), \quad \mathbf{B} = (-2, 0, -4)$$

are linearly dependent since we can take $c_1 = 2, c_2 = 1$, and one of the $c$s is not zero.

Definition 2. A set of vectors $x_1, x_2, \cdots, x_n$ is said to be \textit{linearly independent} if the only solution to

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = \mathbf{0}$$

is the trivial one $c_1 = 0, c_2 = 0, \cdots, c_n = 0$.

For example, the set of vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ is linearly independent.

2. Basis

If a vector has $m$ components, we say that it is an $m-$dimensional vector, or it is in the $m$-dimensional space $\mathbb{R}^m$. Here $m$ and $n$ are positive integers.

Definition 3. If a set of $n$ vectors in $\mathbb{R}^n$ is linearly independent, then it is called to form a \textit{basis}, or simply it is a basis.

Theorem 1. In $\mathbb{R}^n$, any vector $\mathbf{A}$ can be expressed uniquely as a linear combination of a set of basis. That is, let $\mathbf{A}$ be an arbitrary vector in $\mathbb{R}^n$ and $x_1, x_2, \cdots, x_n$ be linearly independent, then there exists a unique set of numbers $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{A} = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$  

To find the coefficients $c_1, c_2, \cdots, c_n$, one can solve the above system of equations.

Theorem 2. In the $m$ dimensional space $\mathbb{R}^m$, any $m + 1$ vectors or more are linearly dependent. This is because a homogeneous system of $m$ linear equations with $m + 1$ or more unknowns always has nonzero solutions.