M597K: Hints to Homework Assignment 7

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1. To show a sequence \( \{x_n\} \) is a Cauchy sequence, one needs to form the difference \(|x_n - x_m|\) and estimate the difference. By using a series of inequalities one can arrive at

\[ |x_n - x_m| < \frac{1}{n^2} \]

for all \( m > n \). See my assistant Xiaoqiang Wang in his office at 217 Osmond Building if you have difficulties in that derivation. His e-mail address is fullheart@etang.com

Then let \( N \) be such that \( \frac{1}{N^2} = \epsilon \), i.e., \( N = \frac{1}{\sqrt{\epsilon}} \). Then for any \( \epsilon > 0 \), choose \( N = \frac{1}{\sqrt{\epsilon}} \). Then for any \( m > n > N \), there holds

\[ |x_n - x_m| < \epsilon. \]

For the technical inequality, one can use

\[
\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\
\leq \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \cdots + \text{ to } \infty \\
\leq \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+1)} + \cdots + \text{ to } \infty \\
\leq \frac{1}{n(n+1)}(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots) \\
\text{use sum of geometric series.} \\
\leq \frac{1}{n(n+1)} \cdot \frac{1}{1 - \frac{1}{n+1}} \\
= \frac{1}{n^2}. \]

There are many other ways to estimate the above sum.

2. Use the set of functions \( 1, x, x^2, x^3, \ldots \). It is infinitely long. None of them is a linear combination of a finite number of others. Remember that two functions \( f(x) \) and \( g(x) \) are equal only if they are identical on \([0, 1]\). Even if they are equal on a finite number of points, they are not the same function provided that they differ on at least one point of the domain.

Also remember that a polynomial of degree \( n \) has at most \( n \) roots.

3. No hint.

4. For the uniform case, the proof is straightforward.
For the $L^2$ norm, use a sequence of functions that approximate the delta function $\delta$.

5. Use the Cauchy-Schwarz inequality:

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} \sqrt{\sum_{i=1}^{\infty} y_i^2}$$

for showing the well defined-ness (i.e., $\langle x, y \rangle < \infty$).

6. No hint.

7. The following example might help understand $L^1$ convergence. This convergence involves the smallness of the domain on which the functions might not vanish.

Example 1. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to the zero function $f(x) = 0$, where

$$f_n(x) = 1 \quad \text{for } x \in (0, \frac{1}{n}); \quad f_n(x) = 0 \text{ for all other } x$$

in the norm $L^1[-1, 1]$.

Proof. We calculate the norm

$$\|f_n - f\|_{L^1[-1, 1]} = \int_{-1}^{1} |f_n(x) - f(x)| \, dx = \int_{-1}^{1} f_n(x) \, dx = \int_{0}^{\frac{1}{n}} 1 \, dx = \frac{1}{n}$$

which is smaller than any given $\epsilon > 0$ for all $n > N \equiv \frac{1}{\epsilon}$.

Back to our homework problem. To show it is not a Cauchy sequence in $C[0,1]$, one needs to find a number $\epsilon_0$ and a pair of sequences $f_n$ and $f_m$ such that

$$\|f_n - f_m\|_{C^0} > \epsilon_0$$

I suggest to use $m = 2n$ and calculate the difference $f_n(t) - f_m(t)$ at the point $t = \frac{1}{2} - \frac{1}{m}$. It should be $\frac{1}{4}$. So take $\epsilon_0 = \frac{1}{8}$. Then for this $\epsilon_0$, for the entire pair $f_n$ and $f_{2n}$ the distance of $f_n$ and $f_{2n}$ measured in the maximum norm is at least $\frac{1}{4}$ for all $n \to \infty$.

8. Use definition, the maximum norm on $f$, and $\frac{1}{\sqrt{2}}$ is integrable.

9. It is similar to the functional $\delta(x)$. For linearity verify that

$$T(\alpha f(x) + \beta g(x)) = \alpha T f(x) + \beta T g(x).$$

Review the lecture note on boundedness.

10. Review the example in the text book, replace its $k(x,y)$ by $k(x,y)w(x)$.

11. Direct plugging.