1. Recall that the product of two \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) is defined as the matrix \( AB = (c_{ij}) \) where
\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (i, j = 1, 2, \ldots, n).
\]
Thus show that
\[
\begin{bmatrix}
\frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\
\frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\
\frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\
\frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\
\frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (1)

Here \((x, y, z)\) represents cartesian coordinates and \((u_1, u_2, u_3)\) represents curvilinear coordinates whose Jacobian is not zero. (From this equation, and the rule

\[
\det(AB) = \det(A) \det(B),
\]

one can easily deduce that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the (forward) transformation: i.e., identity (4) in Section 1.15, Lecture 12.)

Solution. (10 points for each problem) We know that

\[
\frac{\partial u_k}{\partial u_j} = \delta_{kj}
\]

because \(u_1, u_2, \) and \(u_3\) are independent variables. Using the chain rule, we obtain that

\[
\frac{\partial u_k}{\partial u_j} = \frac{\partial u_k}{\partial x_i} \frac{\partial x_i}{\partial u_j} = \frac{\partial u_k}{\partial x_1} \frac{\partial x_1}{\partial u_j} + \frac{\partial u_k}{\partial x_2} \frac{\partial x_2}{\partial u_j} + \frac{\partial u_k}{\partial x_3} \frac{\partial x_3}{\partial u_j}.
\]

Thus

\[
\frac{\partial u_k}{\partial x_i} \frac{\partial x_i}{\partial u_j} = \delta_{kj},
\]

which is the same as (1).

2. The transformation relating the cartesian coordinates \(x, y, z\) to the elliptic cylindrical coordinates \(u, v, z\) is given by the equations

\[
x = a \cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z
\]
(u ≥ 0, 0 ≤ v < 2π, a > 0 constant).

(a) Show that in the xy-plane a curve u = constant represents an ellipse, while a curve v = constant represents half of one branch of a hyperbola.

Solution. Let u = b. In the xy-plane, we have
\[ x = a \cosh b \cos v, \quad y = a \sinh b \sin v. \]
It can be written as
\[ \left( \frac{x}{(a \cosh b)} \right)^2 + \left( \frac{y}{(a \sinh b)} \right)^2 = 1, \]
which represents an ellipse. It occupies the whole ellipse since both \( \cosh u \) and \( \sinh u \) are nonnegative, and \( v \) goes through the entire circle \([0, 2\pi]\).

Let \( v = c \). Using
\[ \cosh^2 u - \sinh^2 u = \left( \frac{e^u + e^{-u}}{2} \right)^2 - \left( \frac{e^u - e^{-u}}{2} \right)^2 = 1, \]
we find
\[ \left( \frac{x}{(a \cos c)} \right)^2 - \left( \frac{y}{(a \sin c)} \right)^2 = 1. \]
This is a hyperbola. It has two branches; one branch passes through the point \((a \cos c, 0)\) while the other passes through the point \((-a \cos c, 0)\). For \( c \in (0, \pi/4) \), we see that both \( x \) and \( y \) are positive, thus it lies in the first quadrant, and \( c = \) constant represents the half branch in the first quadrant. Similarly, one can find that for \( v \in (\pi/2, \pi) \), the branch is in the second quadrant. And so on.

(b) Sketch each curve on the xy-plane corresponding to the values \( u = 0; v = 0; v = \pi; v = \pi/2; \) respectively.

Solution. \( u = 0 \) is the segment \( y = 0, -a < x < a. \)
\( v = 0 \) is the ray \( y = 0, x > a. \)
\( v = \pi \) is the ray \( y = 0, x < -a. \)
\( v = \pi/2 \) is the ray \( x = 0, y > 0. \)
See figure.
(c) Verify that the new coordinate system is orthogonal.

Solution. Let \( \mathbf{R} = x_i i = a \cosh u \cos v i_1 + a \sinh u \sin v i_2 + z i_3 \). Then

\[
\frac{\partial \mathbf{R}}{\partial u} = a \sinh u \cos v i_1 + a \cosh u \sin v i_2, \quad \frac{\partial \mathbf{R}}{\partial v} = -a \cosh u \sin v i_1 + a \sinh u \cos v i_2,
\]

and \( \frac{\partial \mathbf{R}}{\partial z} = i_3 \). We see that the three vectors are orthogonal pairwise.

(d) Show that the arc length in the new coordinate system is given by

\[
ds^2 = a^2(\cosh^2 u - \cos^2 v)(du^2 + dv^2) + dz^2.
\]

Show. All \( g_{ij} = 0 \) for \( i \neq j \). We find

\[
g_{11} = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial u} = (a \sinh u \cos v)^2 + (a \cosh u \sin v)^2 = a^2(\cosh^2 u - \cos^2 v).
\]

Similarly we find \( g_{22} = g_{11} \) and \( g_{33} = 1 \).

3. Consider the new coordinates \( u, v, w \) defined by

\[
u = x - y, \quad v = y + z, \quad w = x - z
\]

(a) Find the inverse transformation. (Solution: Use Gause elimination to find
\[x = (u + v + w)/2, \quad y = (-u + v + w)/2, \quad z = (u + v - w)/2.\]
(b) Show that the coordinate curves are straight lines.
(Solution: Use linear equation concept)

(c) Show that the coordinate system \((u, v, w)\) is not orthogonal. (Combining (b) and (c), we call this an \textit{oblique} coordinate system.)
(Solution: Show \(g_{12} \neq 0\) for example.)

(d) Show that the \(u, v, w\) coordinate axes are left-handed.
(Solution: \(\partial R / \partial u \cdot (\partial R / \partial v \times \partial R / \partial w) = -1/2 < 0\).)

(e) Find the expression \(ds\) of the arc length in the coordinates \((u, v, w)\).
\[ds^2 = 3(du^2 + dv^2 + dw^2)/4 + (du \, dv + dv \, dw - dw \, du)/2.\]

4. Find the expression of \(\nabla f\) for \(f = xy + z\) in cylindrical coordinate system.

\textit{Solution.} In cylindrical coordinates: \(u_1 = r, u_2 = \theta, u_3 = z\), the function becomes \(f = r^2 \sin \theta \cos \theta + z\). The formula is
\[\nabla = u_1 \frac{\partial}{\partial r} + u_2 \frac{1}{r} \frac{\partial}{\partial \theta} + u_3 \frac{\partial}{\partial z}\]
where
\[u_1 = \cos \theta \hat{i}_1 + \sin \theta \hat{i}_2, \quad u_2 = -\sin \theta \hat{i}_1 + \cos \theta \hat{i}_2, \quad u_2 = \hat{i}_3.\]
Applying \(\nabla\) to the given scalar field \(f\), we find
\[\nabla f = r \sin(2\theta)u_1 + r(\cos^2 \theta - \sin^2 \theta)u_2 + u_3.\]

5. Find \(\text{div} \ F\) in spherical coordinates where
\[F = ru_r + \sin \theta u_\phi + r \cos \theta u_\theta.\]

\textit{Solution.} Recall the distance formula from Lecture 13
\[(ds)^2 = (dr)^2 + (r d\phi)^2 + (r \sin \phi d\theta)^2\]
so that \(h_1 = 1, h_2 = r, h_3 = r \sin \phi\). Recall the \(\text{div}\) formula from Lecture 14
\[\text{div} \ F = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (F_2 h_1 h_3) + \frac{\partial}{\partial u_3} (F_3 h_1 h_2) \right].\]
We have $F_1 = r$, $F_2 = \sin \theta$, $F_3 = r \cos \theta$. Let $u_1 = r$, $u_2 = \phi$, $u_3 = \theta$. Then
\[
\frac{\partial}{\partial u_1} (F_1 h_2 h_3) = 3r^2 \sin \phi, \quad \frac{\partial}{\partial u_2} (F_2 h_1 h_3) = r \sin \phi, \quad \frac{\partial}{\partial u_3} (F_3 h_1 h_2) = -r^2 \sin \theta.
\]
Thus
\[
\text{div } F = \frac{1}{r \sin \phi} [3r \sin \phi + \sin \theta \cos \phi - r \sin \theta].
\]

6. (Optional problem) Find the expression of $\nabla^2 f$ in spherical coordinates where $f(x, y, z) = xy + yz + zx$.

**Solution.** The spherical coordinates are
\[
x = r \sin \phi \cos \theta \\
y = r \sin \phi \sin \theta \\
z = r \cos \phi.
\]
In spherical coordinates, the function $f(x, y, z) = xy + yz + zx$ is
\[
f = \frac{1}{2} r^2 [\sin(2\theta) \sin^2 \phi + \sin \theta \sin(2\phi) + \cos \theta \sin(2\phi)].
\]
We already know that $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \phi$. From Lecture 14 notes we know that $\nabla^2 f = \nabla \cdot \nabla f = \text{div } \nabla f = \Delta f$ has formula
\[
\Delta f = \frac{1}{r_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (h_2 h_3 \frac{\partial f}{\partial u_1}) + \frac{\partial}{\partial u_2} (h_1 h_3 \frac{\partial f}{\partial u_2}) + \frac{\partial}{\partial u_3} (h_1 h_2 \frac{\partial f}{\partial u_3}) \right].
\]
We calculate that
\[
\frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (h_2 h_3 \frac{\partial f}{\partial u_1}) = 3[\sin(2\theta) \sin^2 \phi + \sin \theta \sin(2\phi) + \cos \theta \sin(2\phi)].
\]
\[
\frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (h_1 h_3 \frac{\partial f}{\partial u_2}) = \frac{1}{2 \sin \phi} [\cos \phi[\sin(2\theta) \sin(2\phi) + 2 \sin \theta \cos(2\phi) + 2 \cos \theta \cos(2\phi)] \\
+ \sin \phi[2 \sin(2\theta) \cos(2\phi) - 4 \sin \theta \sin(2\phi) - 4 \cos \theta \sin(2\phi)]].
\]
\[
\frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (h_1 h_2 \frac{\partial f}{\partial u_3}) = -\frac{1}{\sin \phi} [2 \sin(2\theta) \sin \phi + \sin \theta \cos \phi + \cos \theta \cos \phi].
\]
Adding the three terms yields the answer to our question.