1. Recall that the product of two \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) is defined as the matrix \( AB = (c_{ij}) \) where
\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (i, j = 1, 2, \ldots, n).
\]
Thus show that
\[
\begin{pmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\
\frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\
\frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3}
\end{pmatrix}
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(1)

Here \( (x_1, x_2, x_3) \) represents cartesian coordinates and \( (u_1, u_2, u_3) \) represents curvilinear coordinates whose Jacobian is not zero. (From this equation, and the rule
\[
\det(AB) = \det(A) \det(B),
\]
one can easily deduce that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the (forward) transformation: i.e., identity (4) in Section 1.15, Lecture 12.)

2. The transformation relating the cartesian coordinates \( x, y, z \) to the elliptic cylindrical coordinates \( u, v, z \) is given by the equations
\[
x = a \cosh u \cos v, \quad y = a \sinh u \sin v, \quad z = z
\]
\((u \geq 0, 0 \leq v < 2\pi, a > 0 \text{ constant })\).

(a) Show that in the \( xy \)-plane a curve \( u = \text{constant} \) represents an ellipse, while a curve \( v = \text{constant} \) represents half of one branch of a hyperbola.

(b) Sketch each curve on the \( xy \)-plane corresponding to the values \( u = 0; v = 0; v = \pi; v = \pi/2 \); respectively.

(c) Verify that the new coordinate system is orthogonal.

(d) Show that the arc length in the new coordinate system is given by
\[
ds^2 = a^2 (\cosh^2 u - \cos^2 v)(du^2 + dv^2) + dz^2.
\]
3. Consider the new coordinates $u, v, w$ defined by

$$u = x - y, \quad v = y + z, \quad w = x - z$$

(a) Find the inverse transformation.
(b) Show that the coordinate curves are straight lines.
(c) Show that the coordinate system $(u, v, w)$ is not orthogonal. (Combining (b) and (c), we call this an oblique coordinate system.)
(d) Show that the $u, v, w$ coordinate axes are left-handed.
(e) Find the expression $ds$ of the arc length in the coordinates $(u, v, w)$.

4. Find the expression of $\nabla f$ for $f = xy + z$ in cylindrical coordinate system.

5. Find $\text{div } \mathbf{F}$ in spherical coordinates where

$$\mathbf{F} = r \mathbf{u}_r + \sin \theta \mathbf{u}_\phi + r \cos \theta \mathbf{u}_\phi.$$

6. (Optional problem) Find the expression of $\nabla^2 f$ in spherical coordinates where $f(x, y, z) = xy + yz + zx$. 