M597K: Homework Assignment 4

Date: Sept. 18, Wed., 2002. Due Friday Sept 27.

1. Given a scalar \( \Phi(x_1, x_2, x_3) \), does the gradient \( \nabla \Phi = (\partial_{x_1} \Phi, \partial_{x_2} \Phi, \partial_{x_3} \Phi) \) satisfy the law of coordinate transformation for first-order tensors? That is,

\[
\partial'_{x_i} \Phi'(x'_1, x'_2, x'_3) = \alpha_{i'k} \partial_{x_k} \Phi(x_1, x_2, x_3),
\]

Here \( \alpha_{i'k} \) is the coordinate transformation from one rectangular coordinate system \( K \) to another rectangular coordinate system \( K' \). Show your work. But you can skip this homework if you know how to solve the next problem.

2. Given a scalar function \( \Phi = \Phi(x_1, x_2, x_3) \), do the quantities \( \frac{\partial^2 \Phi}{\partial x_i \partial x_k} \) form a tensor? Show your work.

3. The stress tensor at a point has components given by

\[
(p_{ij}) = \begin{pmatrix}
1 & -2 & 2 \\
-2 & 3 & 0 \\
2 & 0 & -1
\end{pmatrix}.
\]

Find the stress vector \( (p_b) \) across an area normal to the unit vector

\[ b = (i_1 - i_2 + i_3)/\sqrt{3}. \]

What is the normal stress across such an area (i.e, the projection \( (p_b \cdot b)b \) of the vector \( p_b \) on to \( b \))? 

4. For the stress tensor given in the previous problem,

(a) What is the total force on a unit disk whose normal is in the positive \( x_2 \) direction?

(b) What is the \( x_3 \) component of the total force on a unit disk whose normal is in the positive \( x_1 \) direction?
5. The unit base vectors $i'_i$ of a new coordinate system $K'$ are given by

\[ i'_1 = \frac{i_2 + i_3}{\sqrt{2}}, \quad i'_2 = \frac{i_1 - i_2 + i_3}{\sqrt{3}}, \quad i'_3 = \frac{2i_1 + i_2 - i_3}{\sqrt{6}}. \]

The stress tensor $p_{ik}$ in the system $K$ is of the form

\[ (p_{ik}) = \begin{pmatrix} p_1 & 0 & p_2 \\ 0 & 0 & 0 \\ p_2 & 0 & p_3 \end{pmatrix}. \]

Find the component $p'_{13}$ of the stress tensor $p'_{lm}$ in $K'$.

6. Let $a_i$, $b_j$, and $c_k$ be the components of three vectors. Verify that the 27 quantities $d_{ijk} = 2a_i b_j c_k$ form a tensor of order 3.

7. Form a scalar by contracting the tensor which is given by the matrix

\[ \begin{pmatrix} -5 & 0 & 1 \\ -1 & 3 & 7 \\ 4 & 8 & 2 \end{pmatrix}. \]

8. Given that

\[ (T_{ik}) = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 3 \\ -4 & 0 & 2 \end{pmatrix}, \quad A = i_1 - i_2 + i_3, \quad B = 2i_1 + i_2 - i_3. \]

Find the inner products $T_{ik}A_i$, $T_{ik}A_k$, and $T_{ik}A_iB_k$. If it is too tedious for you, you may choose $i$ or $k$ to be 2 if that $i$ or $k$ is not a dummy variable. (Note: First index in $T$ is the row number.)

9. Show that the delta function $\delta_{ij} = i_i \cdot i_j$ satisfies the law of transformation for second-order tensors, where $i_1, i_2, i_3$ are the unit vectors of a rectangular coordinate system $K$. This delta tensor is called the unit tensor.

10. (Optional) In nonlinear elasticity, the strain is more accurately defined to be

\[ u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) \]

where $u_i$ are again components of the displacement vector. Show it still satisfies the transformation law for second-order tensors.

(This set of homework looks a lot, but try your best. Some of them are really easy.)