M597K: Solution to Homework 3

Date: Sept. 20, Friday

Solutions 1-2: Omitted.

3. (Summation convention) Expand the terms $A_i B^k C_i$ and $a_{ij} b_j$. Is there a summation in $a_i + b_i$?

Solution: (20 points) $A_i B^k C_i = B^k (A_1 C_1 + A_2 C_2 + A_3 C_3)$.

$a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3$.

There is no summation in $a_i + b_i$ since the repeated index $i$ is not in a product.

4. The new coordinate system $K'$ is obtained by rotating the $i_i$ coordinate system an angle $\theta$ about the $x_3$ axis counterclockwise. Find the coefficients (i.e., $\alpha_{i'j}$ and $\alpha_{j'i}$) in the equations:

$$x_i' = \alpha_{i'j} x_j, \quad x_i = \alpha_{j'i} x'_j.$$ 

Solution: (20 points) Once you draw the three-dimensional graphs of the coordinate unit vectors, you can find

$$i'_1 = \cos \theta i_1 + \sin \theta i_2, \quad i'_2 = -\sin \theta i_1 + \cos \theta i_2, \quad i'_3 = i_3.$$

Then by definition of the $\alpha_{i'j}$ and $\alpha_{j'i}$, one can find

$$x'_1 = \cos \theta x_1 + \sin \theta x_2; \quad x'_2 = -\sin \theta x_1 + \cos \theta x_2; \quad x'_3 = x_3.$$

And by rotating backward, one finds

$$x_1 = \cos \theta x'_1 - \sin \theta x'_2; \quad x_2 = \sin \theta x'_1 + \cos \theta x'_2; \quad x_3 = x'_3.$$

5. The unit base vectors $i'_i$ of a new coordinate system $K'$ are given by

$$i'_1 = \frac{i_2}{\sqrt{3}} + \frac{2i_3}{\sqrt{6}}, \quad i'_2 = \frac{i_1}{\sqrt{2}} - \frac{i_2}{\sqrt{3}} + \frac{i_3}{\sqrt{6}}, \quad i'_3 = \frac{i_1}{\sqrt{2}} + \frac{i_2}{\sqrt{3}} - \frac{i_3}{\sqrt{6}}.$$

Find the coefficients in the equation: $x'_i = \alpha_{i'j} x_j$.

Solution: (20 points) One can use the formula $\alpha_{i'j} = i'_i \cdot i_j$ or use the similarity between the expressions of $i'_i$ and $x'_i$ to draw conclusion.
\[
(\alpha_{i'j}) = \begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}
\end{pmatrix}.
\]

6. Given the transformation of coordinates

\[ x'_i = \alpha_{i'j}x_j \]

where

\[
(\alpha_{i'j}) = \begin{pmatrix}
\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\
-\frac{3}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{3}{2}
\end{pmatrix}.
\]

If a vector \( \mathbf{v} \) has the components \((2, -3, 6)\) with respect to the \(x_i\)-coordinate system, find its components in the \(x'_i\)-system.

**Solution:** (20 points) One can use the formula \( x'_i = \alpha_{i'j}x_j \) or the matrix notation \((\alpha_{i'j})\mathbf{v}^T\) to find \( x' = (4\sqrt{2}, \frac{\sqrt{3}}{3}, -\frac{5\sqrt{3}}{3})\).

7. Given the second-order tensor

\[
(a_{ij}) = \begin{pmatrix}
0 & -1 & 3 \\
1 & 0 & 2 \\
-3 & -2 & 0
\end{pmatrix}.
\]

Find the components \(a'_{21}\) and \(a'_{33}\) of this tensor in the coordinate system \(x'_i\) defined by \( x'_i = \alpha_{i'j}x_j \) where

\[
(\alpha_{i'j}) = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

If you can use computer, then find all the components \(a'_{ij}\). (The formula in matrix notation is \((A') = (\alpha)(A)(\alpha)^T\).)

**Solution:** (20 points) One can use \(a'_{ij} = \alpha_{i'k}\alpha_{j'm}a_{km}\) or the matrix way to find

\[
(a'_{ij}) = \begin{pmatrix}
0 & -1 & 2 \\
1 & 0 & -3 \\
-2 & 3 & 0
\end{pmatrix}.
\]

So \(a'_{21} = 1\) and \(a'_{33} = 0\).