1. A coordinate-independent representation of the gradient is given by

\[ \nabla \phi(x) = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} n(y) \phi(y) \, dS_y \]

where \( V \) is a domain that contains the point \( x \) and \( n \) is the unit exterior normal to \( \partial V \). Either give a proof of this formula or say “Yes, I have read the proof from the text, and understand it.”

2. Consider a rigid body rotating about a fixed point \( O \) with angular velocity \( w \). See Figure 1.6.3. The velocity of a point with position vector \( r \) is given by

\[ \mathbf{v} = w \times \mathbf{r}. \]

![Figure 1.6.3. Curl is twice angular velocity.](image)

Show that

\[
\begin{align*}
\text{curl}_2 \mathbf{v} &= 2w_2 \\
\text{curl}_3 \mathbf{v} &= 2w_3.
\end{align*}
\]

Combined with the calculation for \( \text{curl}_1 \mathbf{v} = 2w_1 \) done in Lecture 6, one can conclude that

\[ \text{curl} \mathbf{v} = 2w. \]

3. (Summation convention) Expand the terms \( A_i B^k C_i \) and \( a_{ij} b_j \). Is there a summation in \( a_i + b_i \)?

4. The new coordinate system \( K' \) is obtained by rotating the \( i \), coordinate system an angle \( \theta \) about the \( x_3 \) axis counterclockwise. Find the coefficients (i.e., \( \alpha_{ij} \) and \( \alpha_{ij}' \)) in the equations:

\[
\begin{align*}
x'_i &= \alpha_{ij} x_j \\
x_i &= \alpha_{ij}' x'_j.
\end{align*}
\]
5. The unit base vectors $\mathbf{i}'_i$ of a new coordinate system $K'$ are given by

\[
\mathbf{i}'_1 = \frac{i_2}{\sqrt{3}} + \frac{2i_3}{\sqrt{6}}, \quad \mathbf{i}'_2 = \frac{i_1}{\sqrt{2}} - \frac{i_2}{\sqrt{3}} + \frac{i_3}{\sqrt{6}}, \quad \mathbf{i}'_3 = \frac{i_1}{\sqrt{2}} + \frac{i_2}{\sqrt{3}} - \frac{i_3}{\sqrt{6}}.
\]

Find the coefficients in the equation: $x'_i = \alpha_{ij} x_j$.

6. Given the transformation of coordinates

\[
x'_i = \alpha_{ij} x_j
\]

where

\[
(\alpha_{ij}) = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
-\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
\frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{3}
\end{pmatrix}.
\]

If a vector $\mathbf{v}$ has the components $(2, -3, 6)$ with respect to the $x_i$-coordinate system, find its components in the $x'_i$-system.

7. Given the second-order tensor

\[
(a_{ij}) = \begin{pmatrix}
0 & -1 & 3 \\
1 & 0 & 2 \\
-3 & -2 & 0
\end{pmatrix}.
\]

Find the components $a'_{21}$ and $a'_{33}$ of this tensor in the coordinate system $x'_i$ defined by $x'_i = \alpha_{ij} x_j$ where

\[
(\alpha_{ij}) = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

If you can use computer, then find all the components $a'_{ij}$. (The formula in matrix notation is $(A') = (\alpha)(A)(\alpha)^T$.)