M597K: Solution to Homework Assignment 10

Date: Nov. 11, Monday; Due Wed. Nov. 20.

1. Use the method of characteristics to derive a solution formula to

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + cu = 0, \quad u(0, x) = g(x),
\]

where \(a\) and \(c\) are constants, \(t > 0\), and \(x \in \mathbb{R}^1\). (Hint: Derive an ordinary differential equation for \(u\) along a characteristic curve \(x = x_0 + at\).)

**Solution.** The characteristic line is given by the equation

\[
x'(t) = a
\]

The solutions are \(x(t) = at + x_0\)

where \(x_0\) is a constant depending on \(x\)

Along the characteristic line the equation is

\[
\frac{du}{dt} = -cu
\]

\[
u = we^{-ct}
\]

where \(w\) is a constant

Determine the constants by the initial value

\[u(0, x(0)) = w = g(x(0)) = g(x_0)\]

Replace \(x_0 = x - ct\) we get solution

\[u(t, x) = g(x - at)e^{-ct}.
\]

2. Use the appropriate formula to find the solution \(u(t, x)\) to

\[
\left\{
\begin{array}{l}
\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \\
u(0, x) = e^{-x^2}, \\
\frac{\partial u}{\partial t}(0, x) = 0.
\end{array}
\right.
\]

**Solution.**

According to D’Alenbert formula:
c = 2, g(x) = e^{-x^2}, h(x) = 0, f(t, x) = 0

Therefore the solution is:

\[ u(t, x) = \frac{1}{2} [e^{-(x+2t)^2} + e^{-(x-2t)^2}] + \frac{1}{4} \int_{x-2t}^{x+2t} h(y) dy + \frac{1}{4} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s, y) dy ds \]

\[ u(t, x) = \frac{1}{2} [e^{-(x+2t)^2} + e^{-(x-2t)^2}] . \]

3. Find the value at \( x = 0, t = 2 \) of the solution of the initial value problem

\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, \\
u(0, x) &= 0, \\
\frac{\partial u}{\partial t}(0, x) &= 0.
\end{aligned}
\]

Solution.

According to D’Alembert formula:

\[ c = 1, g(x) = 0, h(x) = 0, f(t, x) = x \]

Therefore the solution is:

\[ u(t, x) = \frac{1}{2} [g(x + ct) + g(x - ct)] + \frac{1}{2} \int_{x-2t}^{x+2t} h(y) dy + \frac{1}{2} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s, y) dy ds \]

\[ u(t, x) = \frac{1}{2} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s, y) dy ds \]

\[ u(t, x) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s, y) dy ds \]

\[ u(t, x) = \frac{1}{2} \int_0^t \int_{t-s}^{t+s} f(s, y) dy ds \]

\[ u(t, x) = \frac{1}{2} \int_0^t \frac{1}{2} [(x + (t-s))^2 - (x + (t-s))^2] ds \]

\[ u(t, x) = \int_0^t x(t-s) ds = xt^2 - \frac{1}{2}xt^2 = \frac{1}{2}xt^2 \]

If \( x = 0, t = 2 \), \( u(2, 0) = 0 \).

4. In the three dimensional homogeneous wave equation \( (f = 0) \) with speed \( c = 1 \), let \( u(0, x_1, x_2, x_3) = 0, \frac{\partial u}{\partial t}(0, x_1, x_2, x_3) = h(x_1, x_2, x_3) \). Suppose that \( h(x_1, x_2, x_3) = 0 \) for \( x_1^2 + x_2^2 + x_3^2 \geq 1 \). Show or explain that \( u(t, 0, 0, 0) = 0 \) for all \( t > 1 \).

Solution.
Using formula
\[ u(t, x_1, x_2, x_3) = \frac{t}{4\pi(ct)^2} \int \int_{|y-x|=ct} h(y) dS_y \]

For this problem
\[ u(t, 0, 0, 0) = \frac{t}{4\pi t^2} \int \int_{|y|=t} h(y) dS_y \]

for \( t > 1 \), \( h(y) = 0 \), for all \( |y| = t \) so the integration is 0.

5. In the two dimensional homogeneous wave equation \((f = 0)\) with speed \( c = 1 \), let
\( u(0, x_1, x_2) = 0 \), \( \frac{\partial u}{\partial t}(0, x_1, x_2) = h(x_1, x_2) \). Suppose that \( h(x_1, x_2) = 0 \) for \( x_1^2 + x_2^2 \geq 1 \)
and \( h(x_1, x_2) > 0 \) for \( x_1^2 + x_2^2 < 1 \). Show or explain that \( u(t, 0, 0) > 0 \) for all \( t > 0 \).

**Solution.**
Using formula
\[ u(t, x_1, x_2) = \frac{1}{2\pi} \int \int_{r<t} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2 \]

For this problem
\[ u(t, 0, 0) = \frac{1}{2\pi} \int \int_{r<t} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2 \]

since for \( t > 1 \), \( h(y) = 0 \), for all \( |y| = t \)
the integral equals to
\[ \frac{1}{2\pi} \int \int_{r<\min(t,1)} \frac{h(y_1, y_2)}{\sqrt{t^2 - r^2}} dy_1 dy_2 \]

since \( h(y) > 0 \), for \( |y| < 1 \) the integral is positive.

**Optional 6.** Use the method of characteristics to solve the problem
\[ \begin{cases} 
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, & t > 0, x \in \mathbb{R}, \\
 u(0, x) = e^x. 
\end{cases} \]

**Solution.**
Characteristic line \( x'(t) = u \), along the characteristic line the equation is \( u'(t) = 0 \). Thus \( u(t) = e^{x_0} \).
\[ x'(t) = u = e^{x_0}, \ x(t) = e^{x_0}t + C. \]
As \( x(0) = x_0 \), we get \( C = x_0 \), \( x = e^{x_0}t + x_0. \)

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As $u = e^{x_0}$, we get $x = ut + \ln u$

$u(x, t)$ is the solution of $ut + \ln u - x = 0$, as there is no other way to express this, we can only write it in this way.

====End====