
6.10.1. Initial Value Problem

Consider
\[\begin{align*}
\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \\
u(t, 0) &= u(t, L) = 0, \\
u(0, x) &= g(x).
\end{align*}\] (1)

We try separation of variables:
\[u(t, x) = \phi(x)G(t).\] (2)

Then
\[\frac{G'(t)}{kG(t)} = \frac{\phi''(x)}{\phi(x)}.\]

Thus we set them to be a common constant \(-\lambda\):
\[G'(t) = -\lambda kG(t),\] (3)
\[\phi''(x) = -\lambda \phi(x).\] (4)

For \(\phi\) satisfying
\[\phi(0) = \phi(L) = 0,\] (5)

we find the solutions to the eigenvalue problem (4)-(5):
\[\phi(x) = c \sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.\] (6)

So we have solutions
\[u(t, x) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right).\] (7)

We need (7) to satisfy the initial condition
\[u(0, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = g(x).\]

By Fourier sine series, we only need to take
\[C_n = \frac{2}{L} \int_{0}^{L} g(x) \sin\left(\frac{n\pi x}{L}\right) dx.\]
So a complete solution to (1) is

\[ u(t, x) = \sum_{n=1}^{\infty} \left( \frac{2}{L} \int_0^L g(x) \sin\left( \frac{n\pi x}{L} \right) dx \right) e^{-k \left( \frac{n\pi}{L} \right)^2 t} \sin\left( \frac{n\pi x}{L} \right). \]

6.10.2. Inhomogeneous Problem

Consider

\[
\begin{align*}
\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} + Q(t, x), & 0 < x < L, \\
\ u(t, 0) &= u(t, L) = 0, \\
\ u(0, x) &= 0.
\end{align*}
\]

Let us use the eigenfunctions and propose a solution in the form

\[ u(t, x) = \sum_{n=1}^{\infty} C_n(t) \sin\left( \frac{n\pi x}{L} \right), \quad C_n(0) = 0. \]

We expand \( Q(t, x) \) as

\[ Q(t, x) = \sum_{n=1}^{\infty} q_n(t) \sin\left( \frac{n\pi x}{L} \right). \]

We note that \( Q(t, x) \) may not have zero value at the boundaries, but the expansion is still valid in the \( L^2(0, L) \) sense. Then (8) can be projected onto the component \( \sin\left( \frac{n\pi x}{L} \right) \):

\[ C_n'(t) = -k \left( \frac{n\pi}{L} \right)^2 C_n + q_n(t). \]

We find

\[ C_n(t) = e^{-k \left( \frac{n\pi}{L} \right)^2 t} \int_0^t e^{k \left( \frac{n\pi}{L} \right)^2 s} q_n(s) ds. \]

Thus a solution to (8) is

\[ u(t, x) = \sum_{n=1}^{\infty} e^{-k \left( \frac{n\pi}{L} \right)^2 t} \int_0^t e^{k \left( \frac{n\pi}{L} \right)^2 s} q_n(s) ds \sin\left( \frac{n\pi x}{L} \right), \]

where

\[ q_n(t) = \frac{2}{L} \int_0^L Q(t, x) \sin\left( \frac{n\pi x}{L} \right) dx. \quad (n = 1, 2, \cdots) \]

6.10.3. Boundary Value Problem

Consider

\[
\begin{align*}
\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \\
u(0, x) &= 0, \\
u(t, 0) &= a(t), \\
u(t, L) &= b(t).
\end{align*}
\]

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We use the variable 
\[ V = u - [a(t) + \frac{x}{L}(b(t) - a(t))]. \]

Then

\[ \begin{align*}
\frac{\partial V}{\partial t} &= k \frac{\partial^2 V}{\partial x^2} - [a'(t) + \frac{x}{L}(b'(t) - a'(t))], \quad 0 < x < L, \\
V(0, x) &= -[a(0) + \frac{x}{L}(b(0) - a(0))], \\
V(t, 0) &= 0, \\
V(t, L) &= 0.
\end{align*} \]

This can be solved by the previous two steps.

We shall solve the heat equation in a two-dimensional rectangular domain off the class, see the lecture notes.

6.10.4. Two-dimensional rectangular domains.

Consider

\[ \begin{align*}
\frac{\partial u}{\partial t} &= k \Delta u, \quad 0 < x < L, \quad 0 < y < H, \\
u(0, x, y) &= g(x, y), \\
u(t, x, y) &= 0 \quad \text{on the boundary of the rectangle.}
\end{align*} \]

We use eigenfunction expansion

\[ u(t, x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn}(t) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right). \]

Then the initial condition

\[ g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{mn}(0) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \]

implies that

\[ C_{mn}(0) = \frac{4}{LH} \int_{0}^{L} \int_{0}^{H} g(x, y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \, dx \, dy \equiv g_{mn}. \]

The equation implies

\[ C_{mn}'(t) = -k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right] C_{mn}. \]

Thus

\[ C_{mn}(t) = g_{mn} e^{-k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right] t}. \]

Hence a solution is

\[ u(t, x, y) = \sum_{n, m=1}^{\infty} g_{mn} e^{-k\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2\right] t} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right). \]