6.6. Laplace and Poisson equations in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Laplace equation:

$$\Delta u = \partial^2_{x_1} u + \partial^2_{x_2} u + \cdots + \partial^2_{x_n} u = 0 \quad \text{in} \quad \mathbb{R}^n.$$  

It describes the temperature distribution in equilibrium (time-independent). Solutions are any linear functions $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ or quadratic functions $x_1^2 - x_2^2$, $x_1 x_2$, or many higher-order polynomials. These solutions are called harmonic functions. But the only bounded solutions are the constant solutions.

Poisson equation:

$$\Delta u = f(\vec{x}) \quad \text{in} \quad \mathbb{R}^n.$$  

We assume that $|f(\vec{x})| \to 0$ as $|x| \to \infty$, and we look for bounded solutions only.

Try Fourier transform:

$$\hat{u}(\vec{\omega}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) e^{i\vec{\omega} \cdot \vec{x}} \, dx.$$  

Then

$$-|\omega|^2 \hat{u} = \hat{f},$$

$$\hat{u} = -\frac{1}{|\omega|} \hat{f}.$$  

Inversion: In $\mathbb{R}^3$ only: (skipped because it needs distribution theory, see next semester)

$$\left(\frac{1}{4\pi|x|}\right)^\ast = \frac{1}{|\omega|^2}.$$  

Convolution formula yields

$$u = -\frac{1}{4\pi|x|} * f.$$  

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, dy \quad \text{in} \quad \mathbb{R}^3 \quad \text{only}.$$  

Another method:

Motivation: We know that

$$\text{div} \vec{E} = q,$$

i.e., the divergence of an electric field is charge density. For the electric potential $A$ such that

$$\vec{E} = \nabla A,$$
the equation

$$\triangle A = q$$

holds. Consider the ideal case where $q$ is a Dirac delta

$$\triangle A = \delta(x).$$

We can look for a spherically symmetric solution

$$A(\vec{x}) = A(|\vec{x}|).$$

We will find the solution, call it $k(x)$. But before finding it, let us see its significance.

We can see by translation that a solution to

$$\triangle U = \delta(x - x_0)$$

is

$$U(x) = k(x - x_0).$$

Further, a solution to

$$\triangle W = f(x_0)\delta(x - x_0)$$  \hspace{1cm} (1)

is

$$W(x) = k(x - x_0)f(x_0).$$

Integrating the $W$ equation (1) in $x_0$, we find

$$\triangle \int_{\mathbb{R}^n} W dx_0 = \int_{\mathbb{R}^n} f(x_0)\delta(x - x_0) dx_0$$

$$= f(x).$$  \hspace{1cm} (2)

Thus

$$u(x) = \int_{\mathbb{R}^n} f(x_0)k(x - x_0) dx_0$$

is a solution to

$$\triangle u = f(x).$$

Thus the general solution to $\triangle u = f(x)$ is given by the convolution of $f(x)$ with the solution $k(x)$ to the simple Poisson equation with a point charge. The solution $k(x)$ is call the fundamental solution. From the previous Fourier transform approach, we know it has the form $k(x) = \frac{1}{4\pi|x|}$ in three space dimensions.

We remark that the integration step (2) is generally known as Superposition.