Announcement: New office hours are
Mondays, Tuesdays, and Wednesdays from 11:00–11:55am.

6.3. (Continued)

For the inhomogeneous problem
\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) = f(t, x_1, x_2, x_3),
\]
(1)
\[u(0, \vec{x}) = 0,\]
(2)
\[\frac{\partial u}{\partial t}(0, \vec{x}) = 0,\]
(3)
a solution is given by Duhamel’s principle (Fritz John, PDE, p.135)
\[
u(t, \vec{x}) = \frac{1}{4\pi c^2} \int_0^t \frac{ds}{t-s} \int_{|y-x|=c(t-s)} f(s, \vec{y})dS_y.
\]
(4)

Duhamel’s principle: Given a time \(t > 0\). Replace force \(f(s, \vec{x}), s \in [0, t]\), by
acquired velocity at time
\[0 = s_1 < s_2 < s_3 < \cdots < s_n < s_{n+1} = t,
\]
and consider \(w_i(s, \vec{x})\):
\[
\frac{\partial^2 w_i}{\partial s^2} - c^2 \left( \frac{\partial^2 w_i}{\partial x_1^2} + \frac{\partial^2 w_i}{\partial x_2^2} + \frac{\partial^2 w_i}{\partial x_3^2} \right) = 0, \quad s > s_i,
\]
(5)
\[w_i(s_i, \vec{x}) = 0,\]
(6)
\[\frac{\partial w_i}{\partial s}(s_i, \vec{x}) = f(s_i, \vec{x})(s_{i+1} - s_i),\]
(7)
The solution \(w_i(s, \vec{x})\), which we assume is zero for \(s < s_i\), is the part of the
displacement \(u(t, \vec{x})\) that is resulted from a pulse force \(f(s, \vec{x})\) during the time interval
\([s_i, s_{i+1}]\), which is equivalent to a velocity \(f(s_i, \vec{x})(s_{i+1} - s_i)\). The final total displace-
ment \(u(t, \vec{x})\) is by superposition
\[
u(t, \vec{x}) = \sum_{i=1}^n w_i(t, \vec{x}).
\]
(8)
Let \(n \to \infty\) and all \(s_{i+1} - s_i \to 0\), the approximation becomes exact. We can solve
(5)-(7) just as before (Poisson formula):
\[
w_i(s, \vec{x}) = \frac{1}{4\pi c^2(s-s_i)} \int_{|y-x|=c(s-s_i)} (s_{i+1} - s_i)f(s_i, \vec{y})dS_y, \quad s > s_i.
\]
Details are in John, PDE, p.135.

**Applications:** Maxwell’s equations of electromagnetism \((\vec{E}, \vec{B})\) in vacuum are

\[
\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \left( \frac{\partial^2 \vec{E}}{\partial x_1^2} + \frac{\partial^2 \vec{E}}{\partial x_2^2} + \frac{\partial^2 \vec{E}}{\partial x_3^2} \right) = 0,
\]

\[
\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \left( \frac{\partial^2 \vec{B}}{\partial x_1^2} + \frac{\partial^2 \vec{B}}{\partial x_2^2} + \frac{\partial^2 \vec{B}}{\partial x_3^2} \right) = 0,
\]

where \(c = \left( \frac{1}{\rho} \right)^{1/2}\) is the speed of light in vacuum. We see that the speed of light is lower in air since the density \(\rho\) is higher.

**6.4. Hadamard’s method of descent.**

This section has not been covered in class, please study it yourself.

In \(\mathbb{R}^2\), the wave equation

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) = 0,
\]

\[
u(0, x_1, x_2) = g(x_1, x_2),
\]

\[
\frac{\partial u}{\partial t}(0, x_1, x_2) = h(x_1, x_2),
\]

can be regarded as a problem in \(\mathbb{R}^3\) where \(u(t, x_1, x_2, x_3)\) is independent of the third dimension \(x_3\). In this way, we find that the spherical integrals on the sphere

\[
|\vec{y} - \vec{x}| = \left( (y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^2 \right)^{1/2} = ct
\]

can be changed into top and bottom integrals over the disk

\[
(y_1 - x_1)^2 + (y_2 - x_2)^2 < (ct)^2.
\]

Thus

\[
u(t, x_1, x_2) = \frac{1}{2\pi c} \int \int_{r<ct} \frac{h(y_1, y_2)}{\left( c^2 t^2 - r^2 \right)^{1/2}} dy_1 dy_2 + \frac{\partial}{\partial t} \left[ \frac{1}{2\pi c} \int \int_{r<ct} \frac{g(y_1, y_2)}{\left( c^2 t^2 - r^2 \right)^{1/2}} dy_1 dy_2 \right],
\]

where \(r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\).
Figure 6.4.1. Integrals on a sphere becomes integrals on a disk.

$$y_3^2 = (ct)^2 - r^2$$