6.14 Vibrating membrane in a circular domain

PDE: \[
\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \quad \text{in disk } r < a \quad \text{in } \mathbb{R}^2,
\]
\hspace{1cm} (1)

Boundary condition: \[u(t, a, \theta) = 0, \quad r = a, \quad \theta \in [-\pi, \pi],\]
\hspace{1cm} (2)

Initial condition: \[
u(0, r, \theta) = \alpha(r, \theta), \]
\[
\frac{\partial u}{\partial t}(0, r, \theta) = \beta(r, \theta).
\]
\hspace{1cm} (3)

Using separation of variables
\[
u(t, r, \theta) = \phi(r, \theta)G(t),
\]
\hspace{1cm} (4)

we find
\[
\Delta \phi + \lambda \phi = 0,
\]
\hspace{1cm} (5)

and
\[
\frac{d^2 G}{dt^2} + \lambda c^2 G = 0.
\]
\hspace{1cm} (6)

We need
\[
\phi(a, \theta) = 0.
\]
\hspace{1cm} (7)

This time the domain is not rectangular. We will not have \(\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)\). We try separation of variables again
\[
\phi(r, \theta) = f(r)g(\theta), \quad 0 < r < a, \quad -\pi < \theta < \pi.
\]
\hspace{1cm} (8)

Recall in \(2 - D:\)
\[
\Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}.
\]

Thus (5) becomes
\[
- \frac{1}{g} \frac{d^2 g}{d\theta^2} = \frac{r}{f} \frac{d}{dr} \left( \frac{df}{dr} \right) + \lambda r^2 =: \mu.
\]
\hspace{1cm} (9)

(Comment on \(u_{xx} + u_{yy} + \lambda u = 0:\) \[u \equiv f(x)g(y): \quad \frac{f_{xx}}{f} + \frac{g_{yy}}{g} + \lambda = 0\]
\[
\Rightarrow \frac{f_{xx}}{f} + \lambda = - \frac{g_{yy}}{g} =: \mu.
\]

We see that \(g\) needs to be periodic in \(\theta:\)
\[
g(\pi) = g(-\pi),
\]
\[
\frac{d}{d\theta} g(\pi) = \frac{d}{d\theta} g(-\pi).
\]
\hspace{1cm} (10)
The “irregular” Sturm-Liouville eigenvalue problem

\[ \frac{d^2 g}{d\theta^2} + \mu g = 0, \quad \text{with (10)} \]  

yields

\[ \mu = \mu_m := m^2, \quad m = 0, 1, 2, \ldots. \]  

\[ g = \sin(m\theta) \quad \text{or} \quad \cos(m\theta). \]  

(13)

Thus for \( m = 0 \) there is one eigenfunction \( g = 1 \), but for \( m > 0 \), there are two linearly independent eigenfunctions. These eigenfunctions generate a complete and orthogonal basis for \( L^2[-\pi, \pi] \). This is the full **Fourier series**: any function \( \Gamma(\theta) \) in \( L^2[-\pi, \pi] \) has the expansion

\[ \Gamma(\theta) = \sum_{m=0}^{\infty} [a_m \cos(m\theta) + b_m \sin(m\theta)]. \]  

(14)

(Define \( b_0 = 0 \) for notational convenience.) All right. Now for each \( \mu_m \), we consider equation (9)

\[ \frac{r}{f} \frac{d}{dr} \left( r \frac{df}{dr} \right) + \lambda r^2 = m^2 \]  

(15)

with the natural condition \( |f(0)| < \infty \) and \( f(a) = 0 \) derived from (7); i.e.,

\[ \begin{cases} 
  r(r f')' + (\lambda r^2 - m^2) f = 0, \\
  |f(0)| < \infty, \quad f(a) = 0. 
\end{cases} \]  

(16)

The solution to (16) are given in section 6.13.3 (Bessel’s functions), and they are

\[ \begin{cases} 
  f(r) = f_{mn}(r) := J_m(\sqrt{\lambda_{mn}} r), \\
  \lambda = \lambda_{mn} := (\frac{z_{mn}}{a})^2, \quad n = 1, 2, \ldots. 
\end{cases} \]  

(17)

We have found \( \phi \) for (5)

\[ \phi = \phi_{mn} := J_m(\sqrt{\lambda_{mn}} r)[a_m \cos(m\theta) + b_m \sin(m\theta)]. \]

For the \( G \) function in (6) we find

\[ G(t) = \cos(c\sqrt{\lambda_{mn}} t) \quad \text{or} \quad \sin(c\sqrt{\lambda_{mn}} t). \]
Combining all the factors, we find a general solution formula

\[ u(t, r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} [A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) \cos(c\sqrt{\lambda_{mn}} t) \\
+ B_{mn} J_m(\sqrt{\lambda_{mn}} r) \sin(m\theta) \cos(c\sqrt{\lambda_{mn}} t) \\
+ C_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) \sin(c\sqrt{\lambda_{mn}} t) \\
+ D_{mn} J_m(\sqrt{\lambda_{mn}} r) \sin(m\theta) \sin(c\sqrt{\lambda_{mn}} t)] \]  

(18)

Imposing the initial condition (3) on (18) will determine the coefficients. For example, let \( \beta = 0 \), we find \( C_{mn} = D_{mn} = 0 \). Then

\[ \alpha(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}} r) \sin(m\theta), \]

where

\[ A_{mn} = \frac{\int_0^\alpha \int_0^{2\pi} \alpha(r, \theta) J_m(\sqrt{\lambda_{mn}} r) \cos(m\theta) r \, dr \, d\theta}{\int_0^\alpha \int_0^{2\pi} J_m^2(\sqrt{\lambda_{mn}} r) \cos^2(m\theta) r \, dr \, d\theta}, \]

\[ B_{mn} = \frac{\int_0^\alpha \int_0^{2\pi} \alpha(r, \theta) J_m(\sqrt{\lambda_{mn}} r) \sin(m\theta) r \, dr \, d\theta}{\int_0^\alpha \int_0^{2\pi} J_m^2(\sqrt{\lambda_{mn}} r) \sin^2(m\theta) r \, dr \, d\theta}. \]

We stopped here in class and I mentioned the concept of Green’s function. I feel it is best for you to read the material when I put the Green’s function in another section, see Section 6.15.

Notes. Two dimensional eigenvalue problems.

I give a summary here for all the two dimensional eigenvalue problems that we have encountered. They have appeared in

1. Poisson equation in a rectangle \( \Omega \) (Section 6.9) (and Homework set 14)

\[ \begin{cases} 
\Delta \phi + \lambda \phi &= 0, & \text{in } \Omega \\
\phi &= 0 & \text{on } \partial \Omega.
\end{cases} \]

2. Or in a disk (Section 6.13, Bessel’s functions).

3. Heat flow in a rectangle, Section 6.10.4, to be up-loaded (also in Homework set 14).

4. Wave equation in a rectangle (Section 6.11), disk (Section 6.13).
Appendix: One-dimensional eigenvalue problem

We provide a complete solution to the eigenvalue problem

\[
\begin{aligned}
\frac{d^2 \phi}{dx^2} + \lambda \phi &= 0, \quad 0 < x < L \\
\phi(0) &= \phi(L) = 0.
\end{aligned}
\]

Solution. The objective of the eigenvalue problem is to find both the parameter \( \lambda \) and a nonzero solution \( \phi \). We use the strategy of shooting. First let \( \lambda = 0 \), and see whether we can find a nonzero solution \( \phi \). In this case, the equation becomes \( \phi'' = 0 \). Thus \( \phi = a_1 + a_2 x \). Then the boundary conditions imply that \( a_1 = a_2 = 0 \). Thus we do not have any nonzero solution for \( \lambda = 0 \). Let us now try to find a negative solution of \( \lambda : \lambda = -c^2, \quad c > 0 \). Then the equation becomes

\[ \phi'' - c^2 \phi = 0. \]

We use the guess work

\[ \phi = e^{\alpha x} \]

to find that

\[ \alpha^2 - c^2 = 0. \]

So \( \alpha = \pm c \) and we have the solution

\[ \phi = a_1 e^{cx} + a_2 e^{-cx}. \]

The boundary conditions imply similarly that \( a_1 = a_2 = 0 \). So there is no solution for \( \lambda = -c^2 \). Let us now try \( \lambda = c^2, c > 0 \), and solution of the form \( \phi = e^{\alpha x} \); we find \( \alpha = \pm ic \) and the solutions are

\[ \phi = a_1 \cos(cx) + a_2 \sin(cx). \]

The boundary condition \( \phi(0) = 0 \) implies \( a_1 = 0 \). The boundary condition \( \phi(L) = 0 \) implies

\[ \sin(cL) = 0. \]

So we choose \( c \), such that \( cL = n\pi, \quad n = 1, 2, \ldots \). Thus \( \lambda = \left( \frac{n\pi}{L} \right)^2 \), and the corresponding solutions are

\[ \phi = a_2 \sin\left( \frac{n\pi x}{L} \right). \]