Announcement: The final exam will be cumulative. It is on Dec 16 (Monday) 6:50–8:50pm. Room: Chambers 105 for last names beginning with A–D, or Chambers 123 for last names beginning with E–Z. A mock exam was given last Monday on the web.

There will be no lecture on new materials on Dec 13, Friday. However, I will be in the classroom answering questions. It can be regarded as a review.

The student evaluation form will be done Wed.

We will cover special functions such as Bessel’s functions, vibrating circular membranes, and Green’s functions this week.


6.13.1. Legendre polynomials: (p.167, Keener)

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n] \]

are eigenfunctions for the differential equation

\[ ((1 - x^2)u')' + \lambda u = 0, \quad -1 < x < 1, \]

\[ \lambda_n = n(n + 1). \]

Boundary condition is that \( u \) is bounded on \(-1 \leq x \leq 1\). The function \( p(x) = 1 - x^2 \) vanishes at \( |x| = 1 \), so it is not regular. (What we covered last time in 6.12 is called a regular Sturm-Liouville eigenvalue problem. But we claim the polynomials are orthogonal and complete nonetheless.)

6.13.2. The Schrödinger equation:

\[ u'' + (E - V(x))u = 0, \quad x \in \mathbb{R}^1. \]

E is eigenvalue and physicist’s notation for \( \lambda \). Let

\[ V(x) = x^2 - 1, \quad u = e^{-\frac{x^2}{2}}. \]

Then

\[ w'' - 2xw' + \lambda w = 0, \]

\[ w = H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad \lambda_n = 2n. \]
The functions \( H_n(x) \) are called the **Hermite polynomials**, which are orthogonal polynomials.

See p.167, Keener's for more special functions.

**6.13.3. Special functions, Bessel’s differential equations.**

Bessel’s differential equation of order \( m \) \((m \geq 0)\) is

\[
z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - m^2)u = 0, \quad 0 < z < \infty.
\]

We state without proof the following. It has two solutions \( J_m(z) \) and \( Y_m(z) \): As \( z \to 0 \), they satisfy

\[
J_m(z) \sim \begin{cases} 
1, & m = 0, \\
\frac{1}{2^mm!}z^m, & m > 0;
\end{cases}
\]

\[
Y_m(z) \sim \begin{cases} 
\frac{2}{\pi} \ln z, & m = 0, \\
-\frac{2^m(m-1)!}{\pi} z^{-m}, & m > 0.
\end{cases}
\]

\( J_m(z) \) are called Bessel functions of the **first kind**, while \( Y_m(z) \) are called the **second kind**. \( J_m \) are bounded, \( Y_m \) are not bounded. Both are oscillatory and decay to zero as \( z \to \infty \). See Figures 16.3.1-2. We shall let \( z_{mn} \) denote the positive zeros of \( J_m(z) \), \( n = 1, 2, \cdots \).

![Figure 6.13.1. Bessel functions of the first kinds.](image-url)
Another form of Bessel’s equation: Let \( z = \sqrt{\lambda}r \), where \( \lambda > 0 \) is a parameter, then the Bessel’s equation becomes

\[
\frac{r^2 d^2 u}{dr^2} + \frac{r du}{dr} + (\lambda r^2 - m^2)u = 0.
\]

Or

\[
(ru')' - \frac{m^2}{r}u + \lambda ru = 0, \quad 0 < r < \infty.
\]

An eigenvalue problem: We propose to study the eigenvalue problem

\[
\begin{cases}
(ru')' - \frac{m^2}{r}u + \lambda ru = 0, & 0 < r < a, \\
|u(0)| < \infty, & u(a) = 0,
\end{cases}
\]

where \( a \) is any positive given number. Solution to the equation with \( |u(0)| < \infty \) are

\[
u(r) = cJ_m(\sqrt{\lambda}r).
\]

Imposing the other boundary condition \( u(a) = 0 \), we have

\[
\sqrt{\lambda}a = z_{mn}.
\]

Thus

\[
\lambda = \lambda_{mn} := \left(\frac{z_{mn}}{a}\right)^2, \quad n = 1, 2, \ldots.
\]
The eigenvalue problem is a singular Sturm-Liouville eigenvalue problem, we claim that orthogonality and completeness both hold:

$$\int_0^a J_m(\sqrt{\lambda_{mn}} r) J_m(\sqrt{\lambda_{mk}} r) r \, dr = 0, \quad n \neq k,$$

and any $\alpha(r)$, such that $\int_0^a r \alpha^2(r) \, dr < \infty$, has expansion

$$\alpha(r) = \sum_{n=1}^{\infty} b_n J_m(\sqrt{\lambda_{mn}} r),$$

where

$$b_n = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}} r) r \, dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}} r) r \, dr}.$$ 

This expansion is called the \textbf{Fourier-Bessel series}. 