5.4. Stability of first-order linear system

Motivation: A solution needs to be stable in order to be useful in practice. The U.S. missile defense system is not yet stable.

Consider

\[ \frac{d\bar{x}}{dt} = A\bar{x}, \quad \bar{x}(0) = \bar{C}. \]  

where \( A \) is an \( n \times n \) matrix of constants, with \( n \) distinct eigenvalues. The solution formula is

\[ \bar{x}(t) = \alpha_1 \tilde{a}_1 e^{\lambda_1 t} + \alpha_2 \tilde{a}_2 e^{\lambda_2 t} + \cdots + \alpha_n \tilde{a}_n e^{\lambda_n t}. \]

**Theorem 1.** If the real parts of all the eigenvalues of the coefficient matrix \( A \) are (strictly) negative, then any solution to (1) goes to zero as \( t \to +\infty \).

**Theorem 2.** If one or more eigenvalues of \( A \) have positive real parts, then some solutions of (1) go to infinity as \( t \to +\infty \).

*Proofs:* They follow from the solution formula if all eigenvalues are distinct. Otherwise, solutions are like \( t^m e^{\lambda t} \) which also go to zero if the real part of \( \lambda \) is negative, or go to infinity if the real part of \( \lambda \) is positive.

Now let us consider a perturbation of (1)

\[ \frac{d\bar{x}}{dt} = A\bar{x} + R(t, \bar{x}). \]  

Suppose

\[ \|R(t, \bar{x})\| \leq \alpha\|\bar{x}\|, \quad \text{on } \{t \geq 0, \|\bar{x}\| < H\}. \]

for some constants \( \alpha \) and \( H > 0 \). Then

**Theorem 3.** If the real parts of all the eigenvalues of \( A \) are (strictly) negative, and (3) holds for a suitably small \( \alpha \), then the zero solution of (1) is asymptotically stable; \( i.e., \) all solutions of (2) with small initial data go to zero as \( t \to +\infty \).

**Theorem 4.** If one or more eigenvalues of \( A \) have positive real parts, then the zero solution is not stable, provided that (3) holds for a suitably small \( \alpha \).

What if one eigenvalue has zero real part and all others have negative real parts? This is called the critical case, and is where bifurcation occurs. We will discuss these issues in the next section. We provide some concrete stability examples below.
Examples 1. Consider

\[
\begin{align*}
\frac{dx_1}{dt} &= \lambda x_1 \\
\frac{dx_2}{dt} &= \mu x_2.
\end{align*}
\]

(4)

Suppose that \(\lambda < 0, \mu < 0\). Then all solutions go to zero as \(t \to +\infty\). Add perturbation \(R(t, x_1, x_2) : \|R(t, \bar{x})\| < \varepsilon \|\bar{x}\|\),

\[
\begin{align*}
\frac{dx_1}{dt} &= \lambda x_1 + R_1(t, x_1, x_2) \\
\frac{dx_2}{dt} &= \mu x_2 + R_2(t, x_1, x_2),
\end{align*}
\]

where \(\varepsilon < \min(|\lambda|, |\mu|)\), the zero solution is stable: all solutions \(x(t) \to 0\) as \(t \to +\infty\).

2. For (4) again, but \(\lambda < 0 < \mu\). Zero is still a solution. But it is not stable since the initially nearby solution

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \begin{pmatrix} 0 \\ \alpha e^{\mu t} \end{pmatrix},
\]

where \(\alpha\) is small, grows to infinity.

3. Consider now for \(\beta \neq 0\), the system

\[
\begin{align*}
\frac{dx_1}{dt} &= \beta x_2 \\
\frac{dx_2}{dt} &= -\beta x_1.
\end{align*}
\]

Differentiating the first equation and using the second equation we find

\[
\frac{d^2x_1}{dt^2} + \beta^2 x_1 = 0.
\]

We can therefore find the solution formula

\[
\begin{align*}
x_1 &= x_1^0 \cos(\beta t) + x_2^0 \sin(\beta t) \\
x_2 &= -x_1^0 \sin(\beta t) + x_2^0 \cos(\beta t).
\end{align*}
\]

Introduce \(\rho(t) = (x_1^2 + x_2^2)^{\frac{1}{2}},\) then \(\rho(t) = ((x_1^0)^2 + (x_2^0)^2)^{\frac{1}{2}}.\) See Figure 5.2 for the phase portrait of the solutions. This solution is however unstable to perturbations of the form

\[
\begin{pmatrix}
0 \\
\alpha x_2
\end{pmatrix}.
\]
where $\alpha > 0$, because then the equation has the matrix
\[
\begin{pmatrix}
0 & \beta \\
-\beta & \alpha
\end{pmatrix},
\]
one of whose eigenvalues has positive real part.

**Notes 1.** The stability of a nonzero solution $w(t)$ can be transformed to the stability of the zero solution to the equation for $v(t) \equiv u(t) - w(t)$.

2. A general nonlinear system
\[
\frac{d\vec{x}}{dt} = \vec{F}(t, \vec{x})
\]
may be approximateed by (2) just as a curve can be approximated by its tangent lines.

![Diagram showing solutions for $\beta < 0$ and $\beta > 0$.](image)

Figure 5.1. Any solution of Example 3 traces a circle.

### 5.5. Hopf bifurcations and example.

Motivation: Bifurcation theory is used in many life sciences, ecological systems, weather system, fluid, chaos, and turbulence.

Consider
\[
\frac{d^2u}{dt^2} + (u^2 - \lambda) \frac{du}{dt} + u = 0. \tag{5}
\]
It has the solution $u = 0$. Let us consider the linearized equation
\[
\frac{d^2u}{dt^2} - \lambda \frac{du}{dt} + u = 0.
\]
When we try solutions of the form \( u = e^{\mu t} \), we find

\[ \mu^2 - \lambda u + 1 = 0. \]

For \( \lambda < 0 \), both roots have negative real parts, so zero solution is stable. For \( \lambda > 0 \), both roots have positive real parts, so zero solution is unstable. At \( \lambda = 0 \), the roots are purely imaginary \( \mu = i, -i \) and the linearized equation has periodic solutions \( u = e^{it} = \cos t + i \sin t \), or \( u = e^{-it} = \cos t - i \sin t \). Both the real and imaginary parts are real solutions \( u(t) = \cos t \) or \( \sin t \).

We write equation (5) in vector form by introducing \( u_1 = u, u_2 = u' \):

\[
\begin{align*}
\dot{u}_1 &= u_2 \\
\dot{u}_2 &= (\lambda - u_1^2)u_2 - u_1.
\end{align*}
\]

Or

\[
\vec{u}(t) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_1^2u_2 \end{pmatrix}.
\]

This nonlinear system has nonzero periodic solution near \( \lambda = 0 \):

\[
\lambda = \frac{\varepsilon^2}{4} + O(\varepsilon^3), \quad u_1(t) = \varepsilon \cos(\omega t) + O(\varepsilon^3), \quad \omega = 1 + O(\varepsilon^3).
\]

We will derive this expansion in perturbation theory next semester. For now we have a bifurcation diagram, see Figure 5.2, and we state a general bifurcation theorem called Hopf bifurcation.
Theorem (Hopf Bifurcation). Suppose the $n \times n$ matrix $A(\lambda)$ has eigenvalues $\mu_j = \mu_j(\lambda), (j = 1, 2, \ldots, n)$, and that for $\lambda = \lambda_0$, $\mu_1(\lambda_0) = i\beta$, $\mu_2(\lambda_0) = -i\beta$ and $\text{Re} \mu_j(\lambda_0) \neq 0$ for all $j > 2$. Suppose further that $\text{Re} (\mu'_1(\lambda_0)) \neq 0$. Then the system of differential equations

$$\frac{du}{dt} = A(\lambda)u + f(u)$$

with $f(0) = 0$, $f(u)$ a smooth function of $u$, has a branch (continuum) of periodic solutions emanating from $u = 0, \lambda = \lambda_0$.

(The direction of bifurcation is not determined by the Hopf Bifurcation Theorem, but must be calculated by a local power series expansion (See Keener)).

We plan to do serious perturbation theory next semester, where we can understand how a mathematician’s perturbation and calculation helps locating the position of the 9th planet of the solar system.
5.6. Another bifurcation example.

See Keener p.478: Nonlinear Eigenvalue problems.

Consider the elastica equation (a.k.a. Euler column)

\[
\begin{cases} 
  y'' + \left( \lambda - \frac{1}{2} \int_0^1 (y')^2 \, ds \right) y = 0 \\
  y(0) = y(1),
\end{cases}
\]

where \( \lambda \) is a parameter.

We see that the integral \( \int_0^1 (y')^2 (s) \, ds \) is a number. So let us introduce the number

\[ \mu = \lambda - \frac{1}{2} \int_0^1 (y')^2 (s) \, ds. \]

Then equation (6) becomes

\[
y'' + \mu y = 0, \quad y(0) = y(1),
\]

which has solutions

\[ y(x) = A \sin(n \pi x), \quad \text{for } \mu = n^2 \pi^2. \]

These solutions produce

\[ \mu = \lambda - \frac{1}{2} \int_0^1 (y')^2 (s) = \lambda - \frac{1}{2} \int_0^1 (An \pi)^2 \cos^2(n \pi x) \, dx = \lambda - \frac{1}{4} (An \pi)^2. \]

To satisfy (6), we need this \( \mu \) to be the same as in (8); that is

\[ \frac{A^2 n}{4} = \frac{\lambda}{n^2 \pi^2} - 1. \]

Thus we find many branches of solutions besides the zero solution, See Figure 5.3.
Figure 5.3. A nonlinear eigenvalue bifurcation diagram.

Each point indicates $y = A \sin (\pi x)$.

A bifurcation branch

$A : \text{Amplitude}$

$bifurcation point$

$n = 1$

$n = 2$

$n = 3$