2.2.4. Taylor series.

We would like to expand an analytic function in a powerful series:

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \]

where

\[ a_n = \frac{f^{(n)}(z_0)}{n!} \]

and \( z_0 \) is a convenient point for an application. Using Cauchy integral formula for derivatives (Corollary 2.4), we find that

\[ a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \]

for any simple contour \( C \) that contains \( z_0 \).

We can use other simple ways to find Taylor series: For example we have

\[ \frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots \]
valid for all \( |z| < 1 \).

Another example is

\[ \frac{z}{1 - z} = z + z^2 + z^3 + \cdots \]

A third example is

\[ \frac{1 + z}{1 - z} = \frac{1}{1 - z} + \frac{z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + \cdots \]

Using these lecture notes together with the text book is recommended.

2.3. Application: Inviscid incompressible steady potential flow.


Take a two-dimensional inviscid fluid of density \( \rho \) and velocity vector \( \mathbf{u} = (u, v) \).

Suppose there is no source and sink, then

\[ \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, d\ell = 0 \]

for any domain \( \Omega \) in the fluid, where \( \mathbf{n} \) is the unit exterior normal to the boundary and \( d\ell \) represents length element. Using the divergence theorem we find

\[ \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, d\ell = \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) \, dx \, dy = 0. \]
Since the domain is arbitrary, we conclude that $\nabla \cdot (\rho \mathbf{u}) = 0$. Assuming that the fluid is incompressible; i.e., $\rho = \text{constant}$, then

$$\nabla \cdot \mathbf{u} = 0.$$ 

If the velocity $\mathbf{u}$ is a gradient of a scalar function $\phi(x, y)$:

$$\mathbf{u} = \nabla \phi,$$

then the flow is called a potential flow. A potential flow is also called irrotational or curl free flow since one can readily check that the curl of $\mathbf{u}$ is zero. In 2-dimensions, the curl is defined as $\text{curl}(u, v) = \partial_x v - \partial_y u$. The scalar function $\phi(x, y)$ is called a potential function (or simply potential) of the flow.

For incompressible and potential flow we have

$$\nabla \cdot \mathbf{u} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \Delta \phi = 0.$$ 

Thus the potential (function) $\phi$ of a potential flow satisfies the Laplace equation for which we have already a solution formula from last lecture.

Another physical quantity that satisfies the Laplace equation $\Delta \phi = 0$ is temperature $\phi$ in a thermal equilibrium.

If we let $f(z) = \phi(x, y) + i\psi(x, y)$ be an analytic function, then both $\phi$ and $\psi$ satisfy the Laplace equation. Also, if we have a $\phi$ that satisfies the Laplace equation, we can always find a $\psi$, for example by

$$\psi(x, y) = \int_C \partial_x \phi \, dy - \partial_y \phi \, dx$$

such that the $f(z) = \phi(x, y) + i\psi(x, y)$ is analytic. Here $C$ can be any path from the origin to the point $(x, y)$. We can use Green’s formula and Cauchy-Riemann conditions to verify those.

The physical interpretation of $\psi$ is very interesting. We can verify that

$$\nabla \phi \cdot \nabla \psi = 0$$

since $\phi_x = \psi_y, \psi_x = -\phi_y$. Thus the level curves of $\phi$ and $\psi$ are orthogonal. Hence the velocity $\mathbf{u}$ points along the tangent of the level curves of $\psi$. Thus the fluid flows along the level curves of $\psi$, which are called streamlines.
An example of such a flow is provided by such a function

\[ f(z) = \left( z + \frac{a^2}{z} \right)V, \]

where \( a > 0 \) and \( V \) are constants. This \( f(z) \) is analytic away from \( z = 0 \). The real part

\[ \phi = (x + \frac{a^2 x}{x^2 + y^2})V \]

is harmonic, whose gradient

\[ \mathbf{u} = \nabla \phi \]

is the velocity field for the flow past a (cross section of) circular cylinder at \(|z| = a\). The imaginary part of \( f(z) \) is

\[ \psi = (y - \frac{a^2 y}{x^2 + y^2})V, \]

whose level curves \( \psi = \text{constant} \) are streamlines which are the dominating lines in Figure 1.

\[ \mathbf{u} = (V, 0) \]

Figure 1. Flow past a cylinder.

Other notes:
1. There is a correction to the solution of HW No. 5, problem 5, page 5. Please download a new copy.
2. There is some addition to Lecture 16. Please download a new version.
3. No new homework assignment this week. HW no. 6 due Wed Oct 16.
4. Exam is this Friday. Mock exam is uploaded. Its answer has a new version.
5. Feedback concerns addressed. There will be more examples in the lecture notes. Applications will be mentioned as much as possible.

===End of Lecture 18, Oct 9====