2.2.3. Cauchy integral formula

We have found that contour integrals of analytic functions are always zero. Only a few integrands with singularities result in nonzero values. The following Cauchy integral formula describes contour integrals extremely well.

**Theorem 2.3.** (Cauchy integral formula) Let $C$ be a simple non-self-intersecting closed curve traversed counterclockwise. Suppose $f(z)$ is analytic everywhere inside $C$. For any point $z$ inside $C$, there holds

$$
\int_C \frac{f(\xi)}{\xi - z} d\xi = 2\pi if(z). \tag{1}
$$

**Proof.** For any $\epsilon > 0$ fixed, we deform the curve $C$ to $C'$ where $C'$ is the circle $|\xi - z| = \epsilon$ such that $|f(z) - f(\xi)| < \epsilon$ for all points $\xi$ inside $C'$. Note that the integrand in (1) $f(z)/(\xi - z)$ is analytic in the region between $C$ and $C'$, we conclude that the integral in (1) is equal to the same integral over $C'$. (This can be achieved by the previous Theorem and a double-sided cut (or bridge) connecting $C$ and $C'$.)

Now on $C'$ we have

$$
\int_C \frac{f(\xi)}{\xi - z} d\xi = \int_{C'} \frac{f(\xi)}{\xi - z} d\xi = f(z) \int_{C'} \frac{1}{\xi - z} d\xi + \int_{C'} \frac{f(\xi) - f(z)}{\xi - z} d\xi = 2\pi if(z) + i \int_0^{2\pi} [f(z + \epsilon e^{i\theta}) - f(z)] d\theta = 2\pi if(z) + i I \tag{2}
$$

where the integral $I$ is such that $|I| \leq 2\pi \epsilon$. Let $\epsilon \to 0$. We recover the Cauchy integral formula. This completes the proof of the theorem.

**Corollary 2.4.** Under the same assumptions of theorem 2.3, there hold

$$
\int_C \frac{f(\xi)}{(\xi - z)^2} d\xi = 2\pi if'(z) \tag{3}
$$

and

$$
n! \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = 2\pi if^{(n)}(z) \tag{4}
$$

for all $n$-th ($n$ a positive integer) order derivatives. And thus analyticity implies that $f(z)$ is infinitely differentiable.

**Corollary 2.5.** (Poisson formula) A solution to the boundary value problem of the Laplacian

$$
\begin{align*}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{in } x^2 + y^2 \leq 1 \\
u(r, \theta) &= u_0(\theta) \quad \text{on the boundary } r = 1
\end{align*} \tag{5}
$$
where \((r, \theta)\) is the polar coordinate and \(u_0(\theta)\) is a given continuous function, is given by the formula

\[
u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta) \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} d\theta.
\]

Proof. Consider an analytic function \(f(z) = u(x, y) + iv(x, y)\) in \(r < 1\). We have the Cauchy-Riemann equations:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
\]

So we have

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}),
\]

thus

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

That is, the real part of an analytic function is a harmonic function (satisfying the Laplace equation). Now we use the Cauchy integral formula

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - z} d\theta \quad \text{(letting } \xi = e^{i\theta})
\]

for \(z\) inside the unit circle and the same formula

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\xi)\xi}{\xi - (\bar{z})^{-1}} d\theta
\]

applied at the point \((\bar{z})^{-1}\) which is outside of the unit circle (\(1/|\xi| > 1\) if \(|z| < 1\).)

Noting that \(\xi = (\bar{z})^{-1}\) on the unit circle, we can add the previous formulas

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \left[ \frac{\xi}{\xi - z} - \frac{1/\xi}{1/|\xi| - 1/|z|} \right] d\theta.
\]

Or

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) \frac{1 - |z|^2}{|\xi - z|^2} d\theta.
\]

Taking the real part of the formula, we obtain Poisson formula. This completes the proof.