1.16. Grad, div, and curl in orthogonal curvilinear coordinate systems.

In this section we derive the expressions of various vector concepts in an orthogonal curvilinear coordinate system.

Let \((u_1, u_2, u_3)\) be such a system:

\[ u_i = \phi_i(x_1, x_2, x_3). \quad (i = 1, 2, 3). \]

Let

\[ x_i = f_i(u_1, u_2, u_3) \]

be the inverse transformation. We introduce the normalized coordinate tangent vectors:

\[ u_i = \frac{1}{h_i} \frac{\partial \mathbf{R}}{\partial u_i} \quad (\text{no summation}) \quad i = 1, 2, 3, \]

where

\[ h_i = \left| \frac{\partial \mathbf{R}}{\partial u_i} \right|. \]

Assume that \((u_1, u_2, u_3)\) is right-handed so that the Jacobian is positive.

1.16.1. Gradient of a scalar field.

Let \(F(x_1, x_2, x_3)\) be a scalar field in a rectangular system. We know that \(\nabla F\) is a vector, which can be represented as a linear combination of any basis. So let

\[ \nabla F = F_1 \mathbf{u}_1 + F_2 \mathbf{u}_2 + F_3 \mathbf{u}_3. \]

We need to find \((F_1, F_2, F_3)\). We recall from Section 1.5.3(of Lecture 5) the coordinate-independent formula

\[ \nabla F(P_0) = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{n} F(y) dS_y \]

where \(V\) is a domain that contains the point \(P_0\) and \(\mathbf{n}\) is the unit exterior normal to \(\partial V\). By the way, we have also the formulas

\[ \nabla \cdot \mathbf{F}(P_0) = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{n} \cdot \mathbf{F}(y) dS_y \]

for the divergence \((\nabla \cdot \cdot)\) of a vector field \(\mathbf{F}\) and and

\[ \nabla \times \mathbf{F}(P_0) = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \mathbf{n} \times \mathbf{F}(y) dS_y \]

for the curl \((\nabla \times)\) which we will use for the representations of div and curl. The three formulas certainly have striking uniformity. Back to our gradient representation, we take \(V\) to be an elementary “curvilinear parallelepiped” of volume

\[ ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3 \]
with faces perpendicular to the coordinate curves, see Figure 1.16.1.

During the lecture I messed up a crucial step. Here is what is needed. The four essential corners of the curvilinear parallelepiped are given approximately by $P_0$, $P_0 + u_1 du_1$, $P_0 + u_2 du_2$, and $P_0 + u_3 du_3$. Although the displacement from $P_0$ to $P_0 + u_1 du_1$ is $u_1 du_1$, the actual (geometric) distance is $ds_1 = h_1 du_1$.

Figure 1.16.1. Curvilinear parallelepiped.

To calculate the surface integral (1), we first note that there are six sides. For the side that passes through $P_0$ and perpendicular to the $u_1$-axis, we have the approximate value

$$-u_1 F(P_0) h_2 du_2 h_3 du_3,$$

where the surface area element is $ds_2 ds_3 = h_2 du_2 h_3 du_3$. The integral on the surface that is parallel to the previous surface is approximately

$$u_1 F(P_0 + du_1 u_1) h_2 du_2 h_3 du_3,$$

where $P_1 = P_0 + du_1 u_1$ is the position of $P_0$ with an increment $du_1$ along the $u_1$-coordinate axis. Combining these two sides and note that the volume of the element is

$$V = ds_1 ds_2 ds_3 = h_1 h_2 h_3 du_1 du_2 du_3,$$

the average becomes

$$\frac{(-F(P_0) + F(P_0 + du_1 u_1)) h_2 h_3 du_2 du_3}{h_1 h_2 h_3 du_1 du_2 du_3} \rightarrow \frac{1}{h_1} \frac{\partial F(P_0)}{\partial u_1} u_1.$$
The del operator has the formula
\[ \nabla = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3}. \]

**Example 1.16a** Find the expression of \( \nabla \) in cylindrical coordinates.

**Solution.** Let \( u_1 = r, u_2 = \theta, u_3 = z \). It is right-handed. We have \( h_1 = 1, h_2 = r, h_3 = 1 \). Also

\[ u_1 = \cos \theta i_1 + \sin \theta i_2, \quad u_2 = -\sin \theta i_1 + \cos \theta i_2, \quad u_3 = i_3. \]

Thus
\[ \nabla = u_1 \frac{\partial}{\partial r} + u_2 \frac{1}{r} \frac{\partial}{\partial \theta} + u_3 \frac{\partial}{\partial z}. \]

**Example 1.16b.** Find the gradient of \( f = xyz \) in the cylindrical coordinates.

**Solution.** We have \( f = r^2z \sin \theta \cos \theta \). Thus
\[ \nabla f = u_1 2rz \sin \theta \cos \theta + u_2 r \cos \theta (\cos^2 \theta - \sin^2 \theta) + u_3 r^2 \sin \theta \cos \theta. \]

1.16.2. **Divergence.** We let
\[ \mathbf{F} = F_1 \mathbf{u}_1 + F_2 \mathbf{u}_2 + F_3 \mathbf{u}_3. \]

Then we can find, similar to the previous section, that
\[ \text{div} \ \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (F_1 h_2 h_3) + \frac{\partial}{\partial u_2} (F_2 h_1 h_3) + \frac{\partial}{\partial u_3} (F_3 h_1 h_2) \right]. \]

**Example 1.16c** Derive the formula for the Laplacian \( \Delta \) defined as \( \Delta = \text{div} \ \nabla \).

**Solution.** Consider an \( \mathbf{F} = \nabla f \). We have
\[ \Delta f = \text{div} \ \nabla f \]
\[ = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( h_1 h_2 \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( h_2 h_3 \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( h_3 h_1 \frac{\partial f}{\partial u_3} \right) \right]. \]

1.16.3. **The curl.**

Similarly, for
\[ \mathbf{F} = F_1 \mathbf{u}_1 + F_2 \mathbf{u}_2 + F_3 \mathbf{u}_3, \]

as \( V \to 0 \). Similarly we can calculate the other four sides. In summary, we find
\[ \nabla F = \frac{1}{h_1} \frac{\partial F}{\partial u_1} \mathbf{u}_1 + \frac{1}{h_2} \frac{\partial F}{\partial u_2} \mathbf{u}_2 + \frac{1}{h_3} \frac{\partial F}{\partial u_3} \mathbf{u}_3. \]
we can derive the formula

\[
\text{curl } \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left| \begin{array}{ccc}
    h_1 \mathbf{u}_1 & h_2 \mathbf{u}_2 & h_3 \mathbf{u}_3 \\
    \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\
    F_1 h_1 & F_2 h_2 & F_3 h_3
\end{array} \right|.
\]

Appendix: Useful expressions

I. In cylindrical coordinates

\[
\begin{align*}
    u_1 &= r, \quad u_2 = \theta, \quad u_3 = z \\
    h_1 &= 1, \quad h_2 = r, \quad h_3 = 1,
\end{align*}
\]

there hold

\[
\begin{align*}
    \text{grad } f &= \frac{\partial f}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{u}_\theta + \frac{\partial f}{\partial z} \mathbf{u}_z, \\
    \text{div } \mathbf{A} &= \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}, \\
    \text{curl } \mathbf{A} &= \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \mathbf{u}_r + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \mathbf{u}_\theta + \\
    &\quad + \frac{1}{r} \left( \frac{\partial}{\partial \theta} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \mathbf{u}_z, \\
    \Delta f &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2},
\end{align*}
\]

where

\[
\begin{align*}
    \mathbf{u}_r &= \cos \theta \mathbf{i}_1 + \sin \theta \mathbf{i}_2, \quad \mathbf{u}_\theta = -\sin \theta \mathbf{i}_1 + \cos \theta \mathbf{i}_2, \quad \mathbf{u}_z = \mathbf{i}_3
\end{align*}
\]
is the local orthonormal basis, and \( \mathbf{A} \) has components \( A_r, A_\theta, A_z \) with respect to this basis.

II. In spherical coordinates. See textbook by Borisenko, p174.

=====End of Lecture 14 =====