4.1 Conservation Laws

We consider the initial value problem for equations describing scalar conservation laws,
\[ u_t + (f(u))_x = 0 \text{ on } \mathbb{R} \times [0, \infty) \]
\[ u(x, 0) = \phi(x) \] \tag{4.1}
where \( f \) is of class \( C^1 \). The name “conservation law” refers to the following fact. If \( u(x, t) \) is a density at the point \( x \) at time \( t \), then, it follows from (4.1) and the fundamental theorem of calculus that the total mass \( \int_\mathbb{R} u(x, t) \, dx \) satisfies,
\[ \frac{d}{dt} \int_\mathbb{R} u(x, t) \, dx = \int_\mathbb{R} u_t \, dx = \int_\mathbb{R} f(u)_x \, dx = \lim_{x \to \infty} f(u(x)) - \lim_{x \to -\infty} f(u(x)). \]
If \( f \) and \( u \) have appropriate decay as \( |x| \to \infty \), the right-hand side is equal to 0. Thus, the total mass is conserved.

Since we assumed that \( f \) is of class \( C^1 \), by the chain rule \((f(u))_x = f'(u)u_x\) and the equation (4.1) becomes
\[ u_t + f'(u)u_x = 0 \text{ on } \mathbb{R} \times [0, \infty) \]
\[ u(x, 0) = \phi(x) \] \tag{4.2}
A solution \( u \) of (4.2) is called a strong solution (or a classical solution) if \( u \) is of class \( C^1 \) and it satisfies (4.1). In the next section we define the notion of a weak solution.

To find a solution \( u \) we use the method of characteristics. Abbreviate \( a(u) = f'(u) \). The characteristic curves satisfying the initial condition are solutions of the following system
\[ t'(\tau, s) = 1, \quad x'(\tau, s) = a(z(\tau, s)), \quad z'(\tau, s) = 0 \]
with the initial conditions
\[ t(0, s) = 0, \quad x(0, s) = s, \quad z(0, s) = \phi(s). \]
Integrating we find that
\[ t(\tau, s) = \tau, \quad x(\tau, s) = a(\phi(s))\tau + s, \quad z(\tau, s) = \phi(s). \]
Hence the characteristics are straight lines \( x = a(\phi(s))t + s \) emanating from the point \((s, 0)\) on the \((x, t)\)-plane. If \( a(\phi(s)) = 0 \), then it is a vertical line and otherwise its slope is \( 1/a(\phi(s)) \). Since \( z' = 0 \) along characteristics, the
solution \( u \) is constant, \( u(x, t) = \phi(s) \) if \( x = a(\phi(s))t + s \). Now assume that \( s \mapsto a(\phi(s)) \) is increasing. This means that the slopes of the characteristics are decreasing and any two characteristics do not intersect. In this case the solution \( u \) is defined for all \( t > 0 \).

However, if there are two points \((s_1, 0)\) and \((s_2, 0)\) on the \( x \)-axis such that \( s_1 < s_2 \) and \( a(\phi(s_1)) > a(\phi(s_2)) \), then the lines \( x = a(\phi(s_1))t + s_1 \) and \( x = a(\phi(s_2))t + s_2 \) intersect, say at \((x_0, t_0)\). But since \( u \) is constant along the characteristics, then \( \phi(s_1) = u(x_0, t_0) \) and also \( \psi(s_2) = u(x_0, t_0) \). Therefore, the solution exists for \( 0 < t \) small but not for all \( t > 0 \). To overcome this problem we define a weaker notion of a solution of (4.1).

### 4.1.1 Weak solutions

**Definition 4.1.** A function \( u : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) is a weak solution of the initial value problem (4.1) if for every smooth function \( v : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) having compact support in \( \mathbb{R} \times [0, \infty) \) the following identity holds,

\[
\int_{[0, \infty) \times \mathbb{R}} [uv_t + f(u)v_x] \, dx dt + \int_{\mathbb{R}} \phi(x)v(x, 0) \, dx = 0 \tag{4.3}
\]

Observe that a function \( u \) does not have to differentiable or even continuous in order to satisfy this definition.

**Proposition 4.2.** Assume that \( u \) is a classical solution of (4.1). Then \( u \) is a weak solution of (4.1).

**Proof.** Take any smooth function \( v \) having compact support in \( \mathbb{R} \times [0, \infty) \). This means that \( v \) vanishes outside of some rectangle \([-a, a] \times [0, b]\). Multiplying (4.1) by \( v \) and integrating over \( \mathbb{R} \times [0, \infty) \), one gets

\[
0 = \int_{\mathbb{R} \times [0, \infty)} (u_t + (f(u))_x)v \, dx dt = \int_{-a}^{a} \int_{0}^{b} (u_t + (f(u))_x)v \, dx dt
\]
The integral on the right hand side can be computed by integrating by parts,

\[ \int_a^b \int_0^b u_t v \, dx \, dt = \int_a^b \left. uv \right|_0^b \, dx - \int_a^b \int_0^b uv_t \, dx \, dt = -\int_a^b \left. uv(x, 0) \right|_0^b \, dx - \int_a^b \int_0^b uv_t \, dx \, dt \]

and

\[ \int_a^b \int_0^b f(u)_x u_t v \, dx \, dt = \int_0^b \left. uv \right|_a^b \, dx - \int_0^b \int_a^b f(u)v_x \, dx \, dt = -\int_0^b \int_a^b f(u)_x v \, dx \, dt. \]

So,

\[ \int_{[0, \infty) \times \mathbb{R}} [uv_t + f(u)v_x] \, dx \, dt + \int_{\mathbb{R}} \phi(x)v(x, 0) \, dx = 0 \]

as claimed.

Conversely, if \( u \) is a weak solution of (4.1) and it is of class \( C^1 \), then it is a strong solution.

Now, suppose that \( C = \{(c(t), t)| t \geq 0\} \) is a smooth curve in \( \mathbb{R} \times [0, \infty) \). Moreover, assume that \( u \) is a weak solution of (4.1) which has a jump discontinuity across the curve \( C \) In other words at every point \((x_0, t_0) \in C\), the following two limits

\[ u^-(x_0, t_0) = \lim_{(x,t)\to(x_0,t_0), (x,t)\in L} u(x,y) \quad \text{and} \quad u^+(x_0, t_0) = \lim_{(x,t)\to(x_0,t_0), (x,t)\in R} u(x,y) \]

where

\[ L = \{(x, t) \in \mathbb{R} \times [0, \infty)| x < s(t)\} \quad \text{and} \quad R = \{(x, t) \in \mathbb{R} \times [0, \infty)| s(t) < x\}. \]

Finally, assume that the weak solution of (4.1) is smooth on \( L \) and \( R \). Hence \( u \) is a classical solution of (4.1) on either side of \( C \).

**Proposition 4.3.** If \( u \) is a weak solution of (4.1) having a jump discontinuity across \( C \) but otherwise it is smooth, then

\[ f(u^-) - f(u^+) = c'(t)(u^- - u^+). \]
Proof. Take any smooth function \( v \) with compact support in \( \mathbb{R} \times [0, \infty) \) so that \( v(x,0) = 0 \) for all \( x \). Then
\[
0 = \int_{\mathbb{R} \times [0, \infty)} [uv_t + f(u)v_x] \, dx \, dt = \int_L [uv_t + f(u)v_x] \, dx \, dt + \int_R [uv_t + f(u)v_x] \, dx \, dt.
\]
Let us consider each of the integrals on the right hand side. Using the divergence theorem and that \( v \) vanishes outside of some rectangle in \( [-a,a] \times [0,b] \) and \( v(x,0) = 0 \), one finds
\[
\int_L [uv_t + f(u)v_x] \, dx \, dt = \int_C [u^- vn_2 + f(u^-)vn_1] \, ds - \int_L [u_t + f(u)_x]v \, dx \, dt
= \int_C [u^- vn_2 + f(u^-)vn_1] \, ds
\]
(4.4)
since \( u_t + f(u)_x = 0 \) on \( L \). Here \( n_1 \) and \( n_2 \) are the components of the outward normal vector \( n \) to \( C \). Similarly,
\[
\int_R (uv_t + f(u)v_x) \, dx \, dt = -\int_C (u^+ vn_2 + f(u^+)vn_1) \, ds. \tag{4.5}
\]
The minus sign in front of the integral on the right-hand side comes from the fact that this time the outward normal is equal to \(-n = (-n_1, -n_2)\). After adding (4.4) to (4.5) one gets
\[
0 = \int_L (uv_t + f(u)v_x) \, dx \, dt + \int_R (uv_t + f(u)v_x) \, dx \, dt
= \int_C [(u^- vn_2 + f(u^-)vn_1) - (u^+ vn_2 + f(u^+)vn_1)] \, ds
\]

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Since this equality holds for all smooth functions, it follows that

\[ [u^−n_2 + f(u^−)n_1] − [u^+n_2 + f(u^+)n_1] = 0, \]

i.e.

\[ (f(u^-) - f(v^+))n_1 = -(u^- - u^+)n_2. \] (4.6)

The velocity vector \( \frac{df}{dt}(c(t), t) = (c'(t), 1) \) is tangent to the curve \( C \). So, the outward norm to \( C \) is equal to \( n = (n_1, n_2) = \frac{1}{\sqrt{(c'(t))^2 + 1}}(1, -c'(t)) \).

Substituting those values for \( n_1 \) and \( n_2 \) in (4.6), one gets

\[ f(u^-) - f(u^+) = c'(t)(u^- - u^+) \]

as claimed.

Introducing the notation:

• jump of \( u \) across \( C \): \( [u] = u^- - u^+ \),
• jump of \( f(u) \) across \( C \): \( [f(u)] = f(u^-) - f(u^+) \),
• speed of the curve \( C \): \( \sigma = c'(t) \)

Proposition 4.3 says that

\[ [f(u)] = \sigma[u]. \] (4.7)

This is so-called Rankine-Hugoniot jump condition.

**Example 4.4.** In what follows we consider Burger’s equation

\[ u_t + uu_x = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \]

\[ u(x, 0) = \varphi(x). \] (4.8)

With \( f(u) = \frac{1}{2}u^2 \), one sees that (4.8) can be written as

\[ u_t + \left( \frac{1}{2}u^2 \right)_x = 0. \]

A characteristic emanating from the point \( s \) on the \( x \)-axis is a straight line \( x = \phi(s) \cdot t + s \) The solution \( u \) is constant along characteristics so that along the above characteristic \( u \) is equal to \( \varphi(s) \). Now we consider solution \( u \) for various initial condition \( \varphi \).
(A) Consider \( \phi \) defined by

\[
\phi(x) = \begin{cases} 
1 & \text{for } x \leq 0 \\
1 - x & \text{for } 0 < x < 1 \\
0 & \text{for } x \geq 1 
\end{cases}
\]

The characteristics for \( s < 1 \) are straight lines \( x = t + s \) and the solution \( u = 1 \) along each of them. For \( 0 < s < 1 \), \( \phi(s) = 1 - s \) and so, the characteristics are lines \( x = (1 - s)t + s \) and \( u \) along such lines is equal to \( 1 - s = \frac{1}{1-rac{s}{1}} \). For \( s \geq 1 \), the characteristics are vertical lines given by \( x = s \) and along each of these lines \( u = 0 \). See Fig. 3. Note that the characteristics intersect for \( t \geq 1 \) and the classical solution does not exist beyond \( t = 1 \). So we seek a weak solution of (4.8). Assume that \( u \) is such a weak solution. According to Proposition 4.3, \( u \) has to satisfy the Rankine-Hugoniot jump condition,

\[
\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = c'(t)(u^- - u^+).
\]

For \( x \leq 0 \), we should have \( u^- = 1 \) and for \( x \geq 1 \) we should have \( u^+ = 0 \). This, \( c'(t) = 1/2 \) The curve \((c(t), t)\) should contain the point \((1, 1)\). So, after integrating \( c'(t) = 1/2 \) we get

\[
c(t) = \frac{t + 1}{2}
\]

and the weak solution \( u \) for \( t \geq 1 \) is given by

\[
u(x, t) = \begin{cases} 
1 & x < \frac{t+1}{2} \\
0 & x > \frac{t+1}{2} 
\end{cases}
\]
(B) Consider $\phi$ defined by

$$\phi(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 0 & \text{for } x > 0. \end{cases}$$

For $s \leq 0$, the characteristics are straight lines $x = t + s$ and $u$ is equal to 1 along them. For $s > 0$, the characteristics are vertical lines $x = s$ and $u$ is equal to 0 along them. Hence the characteristics intersect and the problem doesn’t have a classical solution. We look for a weak solution $u$. As above

Figure 4: Characteristics for the initial condition (B)

we look for the curve $x = c(t)$ such that $u^- = 1$ to the left of this curve and
$u^+ - 0$ to the right and, in addition, $u$ satisfies

$$\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = c'(t)(u^- - u^+).$$

From this we conclude $c'(t) = \frac{1}{2}$. The curve $(c(t), t)$ should contain the point $(0, 0)$. So, $c(t) = \frac{t}{2}$ and a weak solution is defined by

$$u(x, t) = \begin{cases} 
1 & x < \frac{t}{2} \\
0 & x > \frac{t}{2}.
\end{cases}$$

(C) For the final example consider $\phi$ defined by

$$\phi(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
1 & \text{for } x > 0.
\end{cases}$$

For $s \leq 0$, the characteristics are vertical lines $x = s$ and $u$ is equal to 0 along them, and for $s > 0$, the characteristics are given by $x = t + s$ and $u$ is equal to 1 along them. This time the characteristics do not intersect but they do not specify the solution in the blank region.

To fill the blank space one can decide that the solution should be equal to 0 to the left of the curve $C$ and equal to 1 to the right. Then as in (B) one obtains a weak solution

$$u(x, t) = \begin{cases} 
0, & x < \frac{t}{2} \\
1, & x > \frac{t}{2}.
\end{cases}$$
This solution is discontinuous. However, there is another solution which is even continuous and is defined by

\[ u(x, t) = \begin{cases} 
0, & x < 0 \\
\frac{x}{t}, & 0 \leq x < t \\
1, & t \leq x.
\end{cases} \]

This type of solution which fills the wedge is called a *rarefaction wave*. 

\[ \begin{align*}
\frac{\partial u}{\partial x} &= 0 \\
u &= 1 \\
\frac{\partial u}{\partial t} &= \frac{x}{t} \\
u &= 0 \\
\frac{\partial u}{\partial x} &= \frac{x}{t} \\
u &= 1
\end{align*} \]