3.1 Quasilinear equations

Given an initial curve \( \Gamma(s) = (\tilde{\Gamma}(s), u_0(s)) = (x_0(s), y_0(s), u_0(s)) \) in \( \mathbb{R}^3 \) we are interested in finding a solution of the initial value problem for the first-order quasilinear equation,

\[
\begin{align*}
    a(x, y, u)u_x + b(x, y, u)u_y &= c(x, y, u) \\
    u|_{\tilde{\Gamma}} &= u_0.
\end{align*}
\] (3.1)

As in the case of semilinear equations we define the system of characteristic equations

\[
\begin{align*}
    x'(t, s) &= a(x(t, s), y(t, s), z(t, s)) \\
    y'(t, s) &= b(x(t, s), y(t, s), z(t, s)) \\
    z'(t, s) &= c(x(t, s), y(t, s), z(t, s))
\end{align*}
\] (3.2)

with the initial condition

\[
\begin{align*}
    x(0, s) &= x_0(s) \\
    y(0, s) &= y_0(s) \\
    z(0, s) &= u_0(s).
\end{align*}
\] (3.3)

Assuming that \( a, b \) and \( c \) are \( C^1 \)-function near points of the initial curve and assuming that the initial curve is \( C^1 \), the ODE’s theory guarantees the local solution \((t, s) \mapsto (x(t, s), y(t, s), z(t, s))\) which is of class \( C^1 \). Next we have to express \((t, s)\) in terms of \((x, y)\) by inverting the map \((t, s) \mapsto (x(t, s), y(t, s))\). Denoting the inverse map by \((x, y) \mapsto (t(x, y), s(x, y))\), the solution \( u \) of (3.1) is given by

\[
    u(x, y) = z(t(x, y), s(x, y)).
\]

The inverse function theorem tells us that the map \((t, s) \mapsto (x(t, s), y(t, s))\) is invertible near points \((0, s)\) once the following transversality condition holds,

\[
\begin{bmatrix}
    a & \dot{x}_0 \\
    b & \dot{y}_0
\end{bmatrix} \neq 0,
\] (3.4)

where \( a \) and \( b \) are evaluated at \((x_0(s), y_0(s), u_0(s))\) and the derivatives \( \dot{x}_0 \) and \( \dot{y}_0 \) are taken at \( s \).

**Example 3.1.** Solve the Burger’s equation

\[
\begin{align*}
    u_t + uu_x &= 0 \\
    u(x, 0) &= \phi(x)
\end{align*}
\] (3.5)
The system of the characteristic equations is

\[
\begin{align*}
x'(\tau, s) &= z(\tau, s), \quad x(0, s) = s \\
t'(\tau, s) &= 1, \quad t(0, s) = 0 \\
z'(\tau, s) &= 0, \quad z(0, s) = \phi(s)
\end{align*}
\]

For the solutions we get

\[
t(\tau, s) = \tau, \quad x(\tau, s) = \phi(s) \cdot \tau + s, \quad z(\tau, s) = \phi(s).
\]

(3.6)

We check the transversality condition,

\[
\det \begin{bmatrix}
\phi(s) & 1 \\
1 & 0
\end{bmatrix} = -1.
\]

So the map \((\tau, s) \mapsto (x(\tau, s), t(\tau, s))\) is invertible near point \((0, s)\). Denoting by \((x, t) \mapsto (\tau(x, t), s(x, t))\) the inverse map, we get \(u(x, t) = z(\tau(x, t), s(x, t)) = \phi(s(x, t))\). Letting \(s = s(x, t)\) and \(x = x(\tau, s)\), then (3.6) gives \(s = x - \phi(s) \cdot \tau = x - u \cdot t\). Hence

\[
u(x, t) = \phi(x - u(x, t) \cdot t).
\]

Note that the solution \(u\) is given in the implicit form. Since \(z(\tau, s) = \phi(s)\) for \(\tau \geq 0\), it follows that \(u(x(\tau, s), t(\tau, s)) = \phi(s)\), i.e., along the characteristic \((x(\tau, s), t(\tau, s))\) the function \(u\) has constant value equal to \(\phi(s)\). Also observe that the characteristic \((x(\tau, s), t(\tau, s)) = (\phi(s) \cdot \tau + s, \tau)\) is just a straight line \(x = \phi(s) \cdot t + s\) intersecting the \(x\)-axis at \(x = s\). Assume that the function for some values \(s_1 < s_2\), we have \(h(s_1) > h(s_2)\). Then the two characteristics \(x = \phi(s_1) \cdot t + s_1\) and \(x = \phi(s_2) \cdot t + s_2\) are intersecting at some point \((x_0, t_0)\). Then on one hand we have \(u(x_0, t_0) = \phi(s_1)\) and on the other \(u(x_0, t_0) = \phi(s_2)\). This means that the solution \(u\) is not defined for all \(t \geq 0\). This situation occurs for example when \(\phi\) satisfies \(\phi'(s) < 0\) for all \(s\). On the other hand, when \(\phi(s) \geq 0\), then the characteristics emanating form \(s_1 \neq s_2\) on the \(x\) axis will not intersect for positive values of \(t\). In this case the solution \(u\) is defined globally for all \(t \geq 0\).

4 First-Order Equations-Fully nonlinear equations

We consider fully nonlinear equation of the first order

\[
F(x, y, u, u_x, u_y) = 0
\]

(4.1)
We denote the coordinates of $F$ as $(x, y, z, p, q)$ and assume that $F$ is at least $C^1$. As before we want to find values of $u$ by calculating them along curves $(x(t), y(t))$.

To find $(x(t), y(t))$, we abbreviating $\alpha(t) = (x(t), y(t))$ and assuming that we have a solution $u$ of (4.1) of class $C^2$ we let

$$z(t) = u(\alpha(t)) = u(x(t), y(t)), \quad p(t) = u_x(\alpha(t)), \quad \text{and} \quad q(t) = u_y(\alpha(t)).$$

Then differentiating $p$ and $q$ we get

$$p'(t) = u_{xx}(\alpha(t)) \cdot x'(t) + u_{xy}(\alpha(t)) \cdot y'(t)$$
$$q'(t) = u_{xy}(\alpha(t)) \cdot x'(t) + u_{yy}(\alpha(t)) \cdot y'(t). \quad (4.3)$$

Next we differentiate the equation with respect to $x$ and then with respect to $y$ to get

$$F_x + F_z \cdot u_x + F_p \cdot u_{xx} + F_q \cdot u_{yx} = 0$$
$$F_y + F_z \cdot u_y + F_p \cdot u_{xy} + F_q \cdot u_{yy} = 0$$
so that abbreviating $\gamma(t) = (x(t), y(t), z(t), p(t), q(t))$ we have

$$F_x(\gamma(t)) + F_z(\gamma(t))u_x(\alpha(t)) + F_p(\gamma(t))u_{xx}(\alpha(t)) + F_q(\gamma(t))u_{yxx}(\alpha(t)) = 0$$

and

$$F_y(\gamma(t)) + F_z(\gamma(t))u_y(\alpha(t)) + F_p(\gamma(t))u_{xy}(\alpha(t)) + F_q(\gamma(t))u_{yy}(\alpha(t)) = 0.$$ 

To eliminate the second partial derivatives $u_{xx}, u_{xy}, u_{yy}$ from the last two equations we set

$$x'(t) = F_p(\gamma(t)) \quad \text{and} \quad y'(t) = F_q(\gamma(t)) \quad (4.4)$$

so that, using equation in (4.3), the above equations become

$$F_x(\gamma(t)) + F_z(\gamma(t)) \cdot p(t) + p'(t) = 0$$

and

$$F_y(\gamma(t)) + F_z(\gamma(t)) \cdot u_y(\alpha(t)) + q'(t) = 0.$$ 

Hence the equations for $p$ and $q$ are

$$p'(t) = -F_x(\gamma(t)) - F_z(\gamma(t)) \cdot p(t)$$

$$q'(t) = -F_x(\gamma(t)) - F_z(\gamma(t)) \cdot q(t). \quad (4.5)$$

Finally, differentiating $z(t) = u(\alpha(t))$, we get

$$z'(t) = u_x(\alpha(t)) \cdot x'(t) + u_y(\alpha(t)) \cdot y'(t) = F_p(\gamma(t)) \cdot p(t) + F_q(\gamma(t)) \cdot q(t). \quad (4.6)$$

We have arrived at the system of characteristic equations,

$$x'(t) = F_p(\gamma(t))$$

$$y'(t) = F_q(\gamma(t))$$

$$z'(t) = F_p(\gamma(t)) \cdot p(t) + F_q(\gamma(t)) \cdot q(t) \quad (4.7)$$

$$p'(t) = -F_x(\gamma(t)) - F_z(\gamma(t)) \cdot p(t)$$

$$q'(t) = -F_x(\gamma(t)) - F_z(\gamma(t)) \cdot q(t).$$

To incorporate initial condition we write $x(t, s), y(t, s), z(t, s), p(t, s), q(t, s)$ to denote solutions of (4.7) which satisfy the initial condition

$$x(0, s) = x_0(s), y(0, s) = y_0(s), z(0, s) = u_0(s), p(0, s) = p_0(s), q(0, s) = q_0(s)$$

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at $t = 0$. The initial values for $x, y$ and $z$ are given by the initial curve $\Gamma(s)$. However, the initial values $p_0(s)$ and $q_0(s)$ for $p$ and $q$ have to be found. They can't be prescribed arbitrarily and they must satisfy the following condition. First they satisfy the equation,

$$F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0. \tag{4.8}$$

Further, since $u_0(s) = u(x_0(s), y_0(s))$, we get after differentiating this with respect to $s$ we get

$$\dot{u}_0(s) = u_x(x_0(s), y_0(s)) \cdot \dot{x}_0(s) + u_y(x_0(s), y_0(s)) \cdot \dot{y}_0(s)$$

which shows that $p_0(s)$ and $q_0(s)$ should satisfy

$$p_0(s) \cdot \dot{x}_0(s) + q_0(s) \cdot \dot{y}_0(s) = \dot{u}_0(s). \tag{4.9}$$

If the curves $p_0(s)$ and $q_0(s)$ satisfy (4.8)-(4.9), we say that the initial data $(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s))$ are compatible with the PDE (4.1). Having solve the system of characteristic equation (4.7) with the compatible initial condition, we have to invert the map $(t, s) \mapsto (x(t, s), y(t, s))$ and express $t$ and $s$ as functions of $x$ and $y$, $t = t(x, y)$ and $s = s(x, y)$. If this can be done, then the solution $u$ is given by

$$u(x, y) = z(t(x, y), s(x, y)).$$

The following transversality condition guarantees that the map $(t, s) \mapsto (x(t, s), y(t, s))$ can be inverted,

$$\det \begin{bmatrix} F_p & \dot{x}_0 \\ F_q & \dot{y}_0 \end{bmatrix} \neq 0 \tag{4.10}$$

where $F_p$ and $F_q$ are evaluated at $(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s))$ and $\dot{x}_0, \dot{y}_0$ are taken at $s$. This follows, as in the semilinear and quasilinear case, from the inverse function theorem.

**Example 4.1.** Solve the initial value problem

$$u_x u_y - u = 0, \ (x, y) \in (0, \infty) \times \mathbb{R}, \quad u(0, y) = y^2, \ y \in \mathbb{R}.$$
Here the initial curve \( \Gamma(s) = (0, s, s^2) \) and \( F(x, y, z, p, q) = pq - z \). So the system of characteristic equations is as follows,

\[
\begin{align*}
x' &= q \\
y' &= p \\
z' &= 2pq \\
p' &= p \\
q' &= q 
\end{align*}
\] (4.11)

Here \( x = x(t, s), y = y(t, s), z = z(t, s), p = p(t, s) \) and \( q = q(t, s) \), and \( ' \) stands for the derivative with respect to \( t \). We have to find the initial condition \( p_0(s), q_0(s) \) for \( p(t, s) \) and \( q(t, s) \). The curve \((p_0(s), q_0(s))\) has to satisfy the equations

\[
F(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0 \quad \text{i.e.,} \quad p_0(s)q_0(s) = s^2
\]

and

\[
\frac{d}{ds}u_0(s) = p_0(s)x_0(s) + q_0(s)y_0(s) \quad \text{i.e.,} \quad 2s = q_0(s).
\]

Hence \( q_0(s) = 2s \) and \( p_0(s) = \frac{s}{2} \) and full set of the initial condition for the system of characteristic equations (4.11) is

\[
x(0, s) = 0, y(0, s) = s, z(0, s) = s^2, p(0, s) = \frac{s}{2}, \quad q(0, s) = 2s.
\]

Solving the system for \( p \) and \( q \) we find that

\[
p(t, s) = \frac{s}{2}e^t, \quad q(t, s) = 2se^t.
\]

Then \( z' = 2s^2e^{2t} \) so that

\[
z(t, s) = s^2e^{2t}.
\]

Finally, \( x'(t, s) = 2se^t \) and \( y'(t, s) = \frac{s}{2}e^t \) together with the initial condition give

\[
x(t, s) = 2s[e^t - 1], \quad y(t, s) = \frac{s}{2}[e^t + 1].
\]

Next we have to solve the system \( x = \frac{s}{2}[e^{-1}] \) and \( y = 2s[e^t + 1] \) for \( (t, s) \) in terms of \( (x, y) \). After this the solution \( u \) is given by

\[
u(x, y) = z(t, s) = s^2e^{2t} = (se^t)^2.
\]

So we only have to find \( se^t \). The calculation shows that \( se^t = \frac{x + 4y}{4} \) so that

\[
u(x, y) = \frac{(x + 4y)^2}{16}.
\]
Example 4.2. Find the solution of the eikonal equation \( u_x^2 + u_y^2 = 1 \) satisfying \( u = 0 \) on the circle \( x^2 + y^2 = 1 \).

The initial curve \( \Gamma(s) \) is given by \( \Gamma(s) = (x_0(s), y_0(s), u_0(s)) = (\cos s, \sin s, 0) \). Moreover, let \( F(x, y, z, p, q) = p^2 - q^2 - 1 \). Then if \( u \) is a solution of the eikonal equation, we have \( F(x, y, u, u_x, u_y) = 0 \). Hence the system of characteristic equations is given by

\[
\begin{align*}
x' &= F_p = 2p \\
y' &= F_q = 2q \\
z' &= pF_p + qF_q = 2p^2 + 2q^2 \\
p' &= -F_x - pF_z = 0 \\
q' &= -F_x - qF_z = 0,
\end{align*}
\]

(4.12)

where \( x = x(t, s), y = y(t, s), z = z(t, s), p = p(t, s), \) and \( q = q(t, s) \). The initial condition for solutions \( x, y, \) and \( z \) are

\[
x(0, s) = \cos s, \quad y(0, s) = \sin s, \quad z(0, s) = 0.
\]

If the initial conditions \((p_0(s), q_0(s))\) for \( p \) and \( q \) should satisfy

\[
F(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s)) = 0,
\]

which gives

\[
p_0(s)^2 + q_0(s)^2 = 1
\]

and

\[
u'_0(s) = p_0(s) \cdot x'_0(s) + q_0(s) \cdot y'_0(s),
\]

which gives

\[
0 = p_0(s) \cdot (-\sin s) + q_0(s) \cdot \cos s.
\]

Geometrically, by the first equation the point \((p_0(s), q_0(s))\) lies on the unit circle and the second equation means that the dot product \((p_0(s), q_0(s)) \cdot (-\sin s, \cos s) = 0\) so that the vector \((p_0(s), q_0(s))\) is perpendicular to the vector \((-\sin s, \cos s)\). It follows that there are two sets of solutions \((p_0(s), q_0(s))\), namely,

\[
p_0(s) = \cos s, \quad q_0(s) = \sin s
\]

and

\[
p_0(s) = -\cos s, \quad q_0(s) = -\sin s.
\]

Consider the first case. Then the initial condition for the system \((4.12)\) is

\[
x(0, s) = \cos s, \quad y(0, s) = \sin s, \quad z(0, s) = 0, \quad p_0(s) = \cos s, \quad q_0(s) = \sin s.
\]
Then
\[ p(t, s) = \cos s \quad \text{and} \quad q(t, s) = \sin s. \]

Using these two solutions we get
\[
\begin{align*}
  x(t, s) &= 2t \cos s + \cos s = (2t + 1) \cos s, \\
  y(t, s) &= t \sin s + \sin s = (2t + 1) \sin s, \\
  z(t, s) &= 2t.
\end{align*}
\]

We have to solve the equations \( x = (2t + 1) \cos s \) and \( y = (2t + 1) \sin s \) for \((t, s)\). Note that \( x^2 + y^2 = |2t + 1|^2 \) so that \( |2t + 1| = \sqrt{x^2 + y^2} \) and that \( |2t + 1| = 2t + 1 \) if \( t \geq -1/2 \) and \( |2t + 1| = -2t - 1 \) for \( t \leq -\frac{1}{2} \). Since our solution has to satisfy the initial condition, we must take consider \( t \geq -1/2 \).

So \( 2t + 1 = \sqrt{x^2 + y^2} \). Then
\[
\begin{align*}
  u(x, y) &= z(t(x, y), s(x, y)) = 2t(x, y) = -1 + \sqrt{x^2 + y^2}.
\end{align*}
\]

Let us consider the case \( p_0(s) = -\cos s, q_0(s) = -\sin s \). Then the characteristic curves are
\[
\begin{align*}
  x(t, s) &= -2t \cos s + \cos s = (1 - 2t) \cos s, \\
  y(t, s) &= -2t \sin s + \sin s = (1 - 2t) \sin s, \\
  z(t, s) &= 2t. \\
  p(t, s) &= -\cos s. \\
  q(t, s) &= -\sin s.
\end{align*}
\]

Set \( x = (1 - 2t) \cos s \) and \( y = (1 - 2t) \sin s \). Then \( x^2 + y^2 = (1 - 2t)^2 \) and \( \sqrt{x^2 + y^2} = |1 - 2t| \). Since \( |1 - 2t| = 1 - 2t \) if \( 1/2 \leq t \) and \( |1 - 2t| = -1 + 2t \) for \( 1/2 < t \), and the interval \( t \geq 1/2 \) doesn’t contain \( t = 0 \), we see that \( \sqrt{x^2 + y^2} = 1 - 2t \). So,
\[
\begin{align*}
  u(x, y) &= z(t(x, y), s(x, y)) = 2t(x, y) = 1 - \sqrt{x^2 + y^2}.
\end{align*}
\]