10 Elliptic equations

Sections 7.1, 7.2, 7.3, 7.7.1, An Introduction to Partial Differential Equations, Pinchover and Rubinstein

We consider the two-dimensional Laplace equation on the domain $D$,

$$\Delta u = 0, \quad (x, y) \in D.$$ 

More general equation

$$\Delta u = F, \quad (x, y) \in D$$

is called the Poisson equation.

**Definition 10.1.** (1) The problem defined by the Poisson equation $\Delta u = F$ in $D$ together with the Dirichlet boundary condition

$$u(x, y) = g(x, y) \quad (x, y) \in \partial D,$$

is called the Dirichlet problem.

(2) The problem defined by the Poisson equation $\Delta u = F$ in $D$ together with the Neumann boundary condition

$$\partial_n u(x, y) = g(x, y) \quad (x, y) \in \partial D,$$

where $g$ is a given function on the boundary of $D$, is called the Neumann problem. ($\partial_n u$ stands for the normal derivative of $u$, i.e., the directional derivative of $u$ in the direction of unit outward pointing normal vector $n$, $\partial_n u = \nabla u \cdot n = (u_x, u_y) \cdot (n_1, n_2) = u_x n_1 + u_y n_2$.)

(3) The problem defined by the Poisson equation $\Delta u = F$ in $D$ together with the Robin boundary condition

$$u(x, y) + \alpha(x, y) \partial_n u(x, y) = g(x, y) \quad (x, y) \in \partial D,$$

where $g$ and $\alpha$ are given function defined on the boundary of $D$, is called the Robin problem.

**Proposition 10.2.** Consider the Neumann problem

$$\begin{align*}
\Delta u &= F(x, y), \quad (x, y) \in D \\
\partial_n u(x, y) &= g(x, y), \quad (x, y) \partial D
\end{align*} \tag{10.1}$$

Then

$$\int_{\partial D} g \, ds = \int_D F \, dxdy.$$
Proof. Note that
\[ \nabla \cdot \nabla u = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u \]
so that we can write Poisson’s equation as
\[ \nabla \cdot \nabla u = F, \quad (x, y) \in D. \]
Then, in view of the Gauss’ divergence theorem
\[ \int_D F \, dx \, dy = \int_D \nabla \cdot \nabla u \, dx \, dy = \int_{\partial D} \partial_n u \, ds = \int_{\partial D} g \, ds, \]
as claimed. \[\square\]

**Theorem 10.3 (Weak maximum principle).** Let \( D \) be a bounded planar domain and let \( u \in C^2(D) \cap C(\overline{D}) \) be a harmonic function in \( D \). Then \( u \) attains its maximum at the boundary point.

**Proof.** Start with a continuous function \( v \) in the domain \( D \) which satisfies \( \Delta v > 0 \). We claim that \( v \) does not have a local maximum in \( D \). Indeed, assume that \( v \) attains a local maximum at the point \((x_0, y_0)\) in \( D \). Then \( \nabla v(x_0, y_0) = 0 \). Then we must have \( \Delta v(x_0, y_0) \leq 0 \). If \( v_{xx}(x_0, y_0) > 0 \), then
\[ v_{xx}(x_0, y_0) = \lim_{t \to 0} \frac{v_x(x_0 + t, y_0) - v_x(x_0, y_0)}{t} = \lim_{t \to 0} \frac{v_x(x_0 + t, y_0)}{t} > 0. \]
Hence for \( t > 0 \) and small \( v_x(x_0 + t, y_0) > 0 \) showing that \( t \mapsto v(x_0 + t, y_0) \) is increasing for \( t > 0 \) small. Hence \( v(x_0 + t, y_0) > v(x_0, y_0) \) for \( t > 0 \) small. But this contradict the fact that \( v \) has a local maximum at \((x_0, y_0)\). Hence \( v_{xx}(x_0, y_0) < 0 \). Similarly, \( v_{yy}(x_0, y_0) < 0 \). Therefore, \( \Delta v(x_0, y_0) < 0 \), contradicting \( \Delta v > 0 \) in \( D \).

Now given a harmonic function \( u \) note that \( u \) attains maximum at some point in \( \overline{D} \). Take \( \varepsilon > 0 \) and set \( v(x, y) = u(x, y) + \varepsilon(x^2 + y^2) \). Then
\[ \Delta v(x, y) = \Delta u(x, y) + 4\varepsilon = 4\varepsilon > 0. \]
The the above discussion, \( v \) attains its maximum at some boundary point. Set
\[ M := \max\{u(x, y) \mid (x, y) \in \partial D\} \quad \text{and} \quad L := \max\{x^2 + y^2 \mid (x, y) \in \partial D\}. \]
Then
\[ v(x, y) \leq M + \varepsilon L \quad \text{for all} \quad (x, y) \in \partial D \]
94
and since $v$ attains it maximum at the boundary point we have

$$v(x, y) \leq M + \varepsilon L \quad \text{for all } (x, y) \in \overline{D}.$$ 

Hence

$$u(x, y) = v(x, y) - \varepsilon(x^2 + y^2) \leq v(x, y) \leq M + \varepsilon L$$

for all $(x, y) \in \overline{D}$. Taking $\varepsilon \to 0^+$, we get $u(x, y) \leq M$ for all $(x, y) \in \overline{D}$. ■

**Theorem 10.4.** Consider

$$\Delta u = F(x, y), \quad (x, y) \in D$$

$$u(x, y) = g(x, y), \quad (x, y) \partial D$$

Then the problem has at most one solution.

**Proof.** Assume that $u_1$ and $u_2$ are solution of (10.3). Set $v = u_1 - u_2$. Then $v$ satisfies

$$\Delta v = 0, \quad (x, y) \in D$$

$$v(x, y) = 0, \quad (x, y) \partial D$$

Since $v \equiv 0$ along the boundary, the weak maximum principle implies that $v(x, y) \leq 0$ for all $(x, y) \in \overline{D}$. Next note that $w = -v$ solves $\Delta w = 0$ in $D$ and $w(x, y) = 0$ along the boundary. Applying the weak maximum principle to $w$ we get that $-v(x, y) = w(x, y) \leq 0$ for all $(x, y) \in \overline{D}$. So, $0 \leq v(x, y)$ for all $(x, y) \in \overline{D}$. Consequently, $v \equiv 0$, i.e., $u_1 \equiv u_2$ as claimed. ■

### 10.1 Laplace equation on bounded domains

#### 10.1.1 Rectangles

We study the two-dimensional Laplace equation on a rectangle $R = \{(x, y) \in \mathbb{R}^2|0 < x < a, 0 < y < b\}$,

$$\Delta u = u_{xx} + u_{yy} = 0 \quad (x, y) \in R, \quad (10.4)$$

subject to the Dirichlet boundary condition

$$u(0, y) = f(y), \quad u(a, y) = h(y), \quad 0 < y < b,$n

$$u(x, 0) = g(x), \quad u(x, b) = k(x), \quad 0 < x < a. \quad (10.5)$$

To find the solution $u$ we write $u = u_1 + u_2$ where $u_1$ and $u_2$ are harmonic functions in $\mathbb{R}$ and satisfy the following boundary conditions

$$u_1(0, y) = f(y), \quad u_1(a, y) = h(y), \quad 0 < y < b,$n

$$u_1(x, 0) = 0, \quad u_1(x, b) = 0, \quad 0 < x < a.$$
(hence \(u_1\) satisfies homogeneous boundary conditions along the sides \([0, a] \times \{0\}\) and \([0, a] \times \{b\}\), and

\[
\begin{align*}
  u_2(0, y) &= 0, & u_2(a, y) &= 0, & 0 < y < b, \\
  u_2(x, 0) &= g(x), & u_2(x, b) &= k(x), & 0 < x < a 
\end{align*}
\]

(hence \(u_2\) satisfies homogeneous boundary conditions along the sides \(\{0\} \times [0, b]\) and \(\{a\} \times [0, b]\)).

**Example 10.5.** Consider

\[
\Delta u = u_{xx} + u_{yy} = 0 \quad (x, y) \in \mathbb{R},
\]

subject to the Dirichlet boundary condition

\[
\begin{align*}
  u(0, y) &= 0, & u(a, y) &= 0, & 0 < y < b, \\
  u(x, 0) &= g(x), & u(x, b) &= k(x), & 0 < x < a. 
\end{align*}
\]

Note that \(u\) satisfies homogeneous boundary conditions along the sides \(\{0\} \times [0, 1]\) and \(\{a\} \times [0, 1]\). We use separation of variables method and look for solutions of the form

\[u(x, y) = X(x)Y(y).\]

Inserting into the equations we get

\[X''(x)Y(y) + X(x)Y''(y) = 0.\]

After dividing by \(X(x)Y(y)\), we arrive at

\[\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.\]

Since the left-hand sides depends only on \(x\) and the right-hand side on \(y\), both sides are equal to a constant, say

\[\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda.\]

From this we obtain two equations

\[X'' + \lambda X = 0 \quad (10.8)\]

and

\[Y'' - \lambda Y = 0. \quad (10.9)\]
Note that \( 0 = u(a, y) = X(0)Y(y) = X(a)Y(y) = u(b, y) \) for all \( y \) which implies that \( X(0) = X(a) = 0 \). Hence we have to solve the following eigenvalue problem,

\[
\begin{align*}
X'' + \lambda X &= 0, & 0 < x < b, \\
X(0) &= X(a) = 0. \\
\end{align*}
\] (10.10)

The eigenvalues are positive. Indeed, if \( \lambda \leq 0 \) is an eigenvalue and \( X \) is a corresponding eigenfunction, then multiplying the equation by \( X \) and integrating over \([a, b]\), we get

\[
-\lambda \int_a^0 X^2 \, dx = \int_a^0 X''X \, dx = [X'X]_a^b - \int_a^b (X')^2 \, dx = -\int_a^b (X')^2 \, dx.
\]

The left-hand side is greater or equal to 0 whereas the right-hand side is less or equal to 0. Consequently, both sides are equal to 0. This implies that \( X' = 0 \), i.e., \( X' = \text{constant} \) and since \( X(0) = 0 \) this constant has to be equal to 0. So, the eigenvalues are positive. For \( \lambda > 0 \), the general solution of the equation is given by

\[
X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x.
\]

Using boundary conditions we find that

\[
A = 0 \quad \text{and} \quad B \sin \sqrt{\lambda} a = 0.
\]

So for a nontrivial solution we must have \( \sin \sqrt{\lambda} a = 0 \) which implies that the eigenvalues are

\[
\lambda_n = \left( \frac{n\pi}{b} \right)^2, \quad n \geq 1
\]

and the corresponding eigenvalues are

\[
X_n(x) = \sin \frac{n\pi x}{a}, \quad n \geq 1.
\]

Next we look at the equation (11.7) with \( \lambda = \lambda_n \). The general solution is given by

\[
Y_n(y) = A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a}. \] (10.11)

Hence the product solution \( u_n(x, y) = X_n(x)Y_n(y) \) is equal to

\[
u_n(x, y) = \sin \frac{n\pi x}{a} \left[ A_n \cosh \frac{n\pi y}{a} + B_n \sinh \frac{n\pi y}{a} \right],
\]

97
and a proposed solution \( u \) of Laplace equation is a linear combination of these product solution,

\[
u(x, y) = \sum_{n \geq 1} u_n(x, y) = \sum_{n \geq 1} \sin \frac{n \pi x}{a} \left[ A_n \cosh \frac{n \pi y}{a} + B_n \sinh \frac{n \pi y}{a} \right].
\]

The solution \( u \) satisfy Laplace equation and the homogeneous boundary conditions, \( u(0, y) = u(a, y) = 0 \). At \( y = 0 \),

\[
g(x) = u(x, 0) = \sum_{n \geq 1} A_n \sin \frac{n \pi x}{a}.
\]

The coefficients \( A_n \) are given by

\[
A_n = \frac{2}{a} \int_0^a g(x) \sin \frac{n \pi x}{a} dx.
\]

At \( y = b \),

\[
k(x) = u(x, b) = \sum_{n \geq 1} \left[ A_n \cosh \frac{n \pi b}{a} + B_n \sinh \frac{n \pi b}{a} \right] \sin \frac{n \pi x}{a}
\]

from which we find that

\[
A_n \cosh \frac{n \pi b}{a} + B_n \sinh \frac{n \pi b}{a} = \frac{2}{a} \int_0^a k(x) \sin \frac{n \pi x}{a} dx.
\]

and

\[
B_n = -A_n \tanh \frac{n \pi b}{a} + \frac{2}{a \cosh \frac{n \pi b}{a}} \int_0^a k(x) \sin \frac{n \pi x}{a} dx.
\]

To simplify the calculation one can take for the general solution \( Y_n \) (instead of a linear combination of \( \cosh \frac{n \pi y}{a} \) and \( \sinh \frac{n \pi y}{a} \) as in (11.8)) a linear combination of \( \sinh \frac{n \pi y}{a} \) and \( \sinh \frac{n \pi}{a} (y - b) \) (both are linearly independent solutions of (11.7)), namely

\[
Y_n(y) = C_n \sinh \frac{n \pi y}{a} + D_n \sinh \frac{n \pi}{a} (y - b).
\]

Then

\[
u(x, y) = \sum_{n \geq 1} \sin \frac{n \pi x}{a} \left[ C_n \sinh \frac{n \pi y}{a} + D_n \sinh \frac{n \pi}{a} (y - b) \right].
\]
At $y = 0$, 
\[ g(x) = u(x, 0) = \sum_{n \geq 1} (-D_n) \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} \]
from which we find that 
\[ D_n = \frac{-2}{a \sinh \frac{n\pi b}{a}} \int_0^a g(x) \sin \frac{n\pi x}{a} dx. \]

At $y = b$, 
\[ k(x) = u(x, b) = \sum_{n \geq 1} C_n \sinh \frac{n\pi b}{a} \]
from which we find that 
\[ C_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a g(x) \sin \frac{n\pi x}{a} dx. \]

**Example 10.6.** Consider 
\[ \Delta u = u_{xx} + u_{yy} = 0 \quad 0 < x < \pi, 0 < y < \pi, \quad (10.12) \]
subject to the Neumann boundary condition 
\[ u_y(x, 0) = 0, \quad u_y(x, \pi) = x - \frac{\pi}{2}, \quad 0 < x < \pi, \]
\[ u_x(0, y) = u(\pi, y) = 0, \quad 0 < y < \pi. \quad (10.13) \]

Note that along the boundary of the rectangle $R = \{(x, y)|0 < x, y < \pi\}$,
\[ \int_{\partial R} \partial_n u ds = 0. \]

Indeed, $\partial_n u := \nabla u \cdot u = (u_x, u_y) \cdot (n_1, n_2)$. Along $[0, 1] \times \{0\}$, we have $(u_x, u_y) = (u_x, 0)$ and $(n_1, n_2) = (0, -1)$ so that $\partial_n u = 0$. Along $\{\pi\} \times [0, 1]$, $(u_x, u_y) = (0, u_y)$ and $(n_1, n_2) = (1, 0)$, and $\partial_n u = 0$. Along $[0, 1] \times \{\pi\}$, $(u_x, u_y) = (u_x, x - \pi/2)$ and $(n_1, n_2) = (0, 1)$, and $\partial_n u = x - \pi/2$. Finally, along $\{0\} \times [0, 1]$, $(u_x, u_y) = (0, u_y)$ and $(n_1, n_2) = (-1, 0)$ so that $\partial_n u = 0$. Consequently, 
\[ \int_{\partial R} \partial_n u \, ds = - \int_0^\pi (x - \pi/2) \, dx = 0. \]

Hence the necessary condition for solvability is satisfied. Next we seek a solution $u$ of the form 
\[ u(x, y) = X(x)Y(y). \]
Inserting into the equation and dividing by $XY$ we get

$$\frac{X''(x)}{X} = -\frac{Y''}{Y}.$$  

Since the left-hand side depends only on $x$ and the right-hand side depends only on $y,$

$$\frac{X''(x)}{X} = -\frac{Y''}{Y} = -\lambda$$

for some constant $\lambda.$ Hence $X$ and $Y$ satisfy

$$X''(x) + \lambda X = 0 \quad (10.14)$$

and

$$Y''(x) - \lambda Y = 0. \quad (10.15)$$

Moreover, if $u_x(0, y) = X'(0)Y(y) = 0$ and $u_x(\pi, y) = X'(\pi)Y(y) = 0.$ Since we look for a nontrivial solution, we must have $X'(0) = X'(\pi) = 0.$ So have to study eigenvalue problem

$$X''(x) + \lambda X = 0, \quad 0 < x < \pi$$

$$X'(0) = X'(\pi) = 0,$$ \quad (10.16)

The eigenvalues are

$$\lambda_n = n^2, \quad n \geq 0,$$

and the corresponding eigenfunctions are

$$X_n(x) = \cos nx.$$

Next we solve (11.12) with $\lambda = \lambda_n = n^2,$

$$Y''(x) - n^2 Y = 0.$$  

If $n \geq 1,$ then

$$Y_n(y) = A_n \cosh ny + B_n \sinh ny,$$

and if $n = 0,$ then

$$Y_0 = A_0 + B_0 y.$$  

Note that along the side $[0, 1] \times \{0\}, \ u_y(x, 0) = 0$ so that $X(x)Y'(0) = 0$ showing that we must have $Y'_n(0) = 0$ for every $n \geq 0.$ Since $Y'_0(y) = B_0$ and $Y'_n(y) = nA_n \sinh ny + nB_n \cosh ny.$ Hence $Y'_0(0) = B_0 = 0$ and $Y'_n(0) = nB_n = 0.$ Consequently,

$$Y_n = A_n \cosh ny, \quad n \geq 0.$$
The product solution \( u_n(x, y) = X_n(x)Y_n(y) \) has the form
\[
u_n(x, y) = A_n A_n \cos nx \cosh ny,\]
and a solution \( u \) is a infinite series of product solutions
\[
u(x, y) = \sum_{n \geq 0} u_n(x, y) = \sum_{n \geq 0} A_n \cosh ny \cos nx.\]

Differentiating (formally) with respect to \( y \) we get
\[
u_y(x, y) = \sum_{n \geq 1} nA_n \sinh ny \cos nx.\]
At \( y = \pi \),
\[
x - \pi / 2 = \nu_y(x, \pi) = \sum_{n \geq 1} nA_n \sinh n\pi \cos nx.
\]
Thus,
\[
nA_n \sinh n\pi = \frac{2}{\pi} \int_0^\pi (x - \pi / 2) \cos nx \, dx = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx - \int_0^\pi \cos nx \, dx
\]
\[
= \frac{2}{\pi} \left[ \left. \frac{x \sin nx}{n} \right|_0^\pi - \frac{2}{n\pi} \int_0^\pi \sin nx \, dx \right] - \frac{1}{\pi} \int_0^\pi \cos nx \, dx
\]
\[
= \frac{2}{n^2\pi} \left[ \cos nx \right|_0^\pi - \left[ \frac{\sin nx}{n} \right]_0^\pi
\]
\[
= \frac{2}{n^2\pi} \left[ (-1)^n - 1 \right] = \begin{cases} -\frac{4}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}
\]
This means that \( A_n = 0 \) is \( n \) is even and if \( n \) is odd, then
\[
A_n = -\frac{4}{n^3\pi \sinh n\pi}
\]
so that
\[
u(x, y) = A_0 + \frac{4}{\pi} \sum_{n \geq 1} \frac{\cosh((2n - 1)y) \cos((2n - 1)x)}{(2n - 1)^3 \sinh(2n - 1)\pi}
\]
Finally, we consider a Laplace equation with a mixed boundary conditions.
Example 10.7. Consider

\[ \Delta u = 0, \quad 0 < x < a, 0 < y < b \]
\[ u(0, y) - u_x(0, y) = 0, \quad u(a, y) = f(y), \quad 0 < y < b, \]  \hspace{1em} (10.17)
\[ u(x, 0) = u(x, b) = 0, \quad 0 < x < a. \]

Using separation of variables, we have

\[ \frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \]

I.e.,

\[ X'' - \lambda X = 0 \]
\[ Y'' + \lambda Y = 0 \]

Next note that \( u(x, 0) = u(x, b) = 0 \) so that \( X(x)Y(0) = X(x)Y(b) = 0 \).

This implies that \( Y(0) = Y(b) = 0 \) and we we consider the eigenvalue problem

\[ Y'' + \lambda Y = 0, \quad 0 < y < b \]
\[ Y(0) = Y(b) = 0. \]

The eigenvalues are

\[ \lambda_n = \left( \frac{n\pi}{b} \right)^2, \quad n \geq 1 \]

with the corresponding eigenfunctions

\[ Y_n(y) = \sin \frac{n\pi y}{b}, \quad n \geq 1. \]

Now we need to solve

\[ X'' - \lambda_n X = 0. \]

Along the side \( \{0\} \times [0, 1], \) \( u(0, y) - u_x(0, y) = 0 \) so that

\[ X(0)Y(y) - X'(0)Y(y) = 0 \]

implying that \( X(0) = X'(0) = 0. \) For \( \lambda = \lambda_n = \left( \frac{n\pi}{b} \right)^2, \) the general solution of \( X'' - \lambda_n X = 0 \) is given by

\[ X_n(x) = A_n \cosh \frac{n\pi x}{b} + B_n \sinh \frac{n\pi x}{b}. \]
Since \( X'_1(x) = A_n \frac{n\pi}{b} \sinh \frac{n\pi x}{b} + B_n \frac{n\pi}{b} \cosh \frac{n\pi x}{b} \), the condition \( X(0) = X'(0) = 0 \) implies that
\[
A_n = B_n \frac{n\pi}{b}.
\]
Consequently,
\[
X_n(x) = B_n \frac{n\pi}{b} \cosh \frac{n\pi x}{b} + B_n \sinh \frac{n\pi x}{b}.
\]
Therefore the we look for the solution of (11.14) as a series of product solutions
\[
u(x, y) = \sum_{n \geq 1} B_n \sin \frac{n\pi}{b} \left[ \frac{n\pi}{b} \cosh \frac{n\pi x}{b} + \sinh \frac{n\pi x}{b} \right].
\]
Finally, along \( \{a\} \times [0, 1] \),
\[
f(y) = u(a, y) = \sum_{n \geq 1} B_n \left[ \frac{n\pi}{b} \cosh \frac{n\pi a}{b} + \sinh \frac{n\pi a}{b} \right] \sin \frac{n\pi y}{b}.
\]
Therefore,
\[
B_n \left[ \frac{n\pi}{b} \cosh \frac{n\pi a}{b} + \sinh \frac{n\pi a}{b} \right] = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy,
\]
i.e.,
\[
B_n = \frac{2}{b \left[ \frac{n\pi}{b} \cosh \frac{n\pi a}{b} + \sinh \frac{n\pi a}{b} \right]} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy.
\]