MATH 412 Fourier Series and PDE- Spring 2010

SOLUTIONS to HOMEWORK 3

Problem 1.

(a): Find a general solution of the equation

\[ u_{tt} + 3u_{xt} - 3u_{xx} = 0. \]

(b): Find a solution of the Cauchy problem,

\[ u_{tt} + 3u_{xt} - 4u_{xx} = 0 \quad (x, t) \in \mathbb{R} \times (0, \infty) \]

\[ u(x, 0) = \phi(x) \quad x \in \mathbb{R} \]

\[ u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}. \]

Solution: (b) The discriminant of the equation is positive since \( \det \begin{bmatrix} 3/2 & 1 \\ -4 & 3/2 \end{bmatrix} = 9/4 + 4 = 25/4 > 0 \), hence the equation is hyperbolic. Solve \( \mu_1^2 + 3\mu - 4 = 0 \) to get two real solutions, \( \mu_1 = -4 \) and \( \mu_2 = 1 \). There are two characteristic equations

\[ \frac{dx}{dt} = 4 \quad \text{and} \quad \frac{dx}{dt} = -1 \]

whose solutions are

\[ x - 4t = C_1 \quad \text{and} \quad x + t = C + 2. \]

Define the change of coordinates by \( \xi = x - 4t \) and \( \eta = x + t \). Then

\[ \xi_x = 1, \quad \xi_t = -4, \quad \eta_x = 1, \quad \eta_t = 1. \]

Setting \( u(x, t) = v(\xi, \eta) = v(x - 4t, x + t) \), one finds that

\[
\begin{align*}
  u_x &= v_\xi + v_\eta \\
  u_t &= -4v_\xi + v_\eta \\
  u_{xx} &= v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta} \\
  u_{xt} &= -4v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta} \\
  u_{tt} &= 16v_{\xi\xi} - 3v_{\xi\eta} + v_{\eta\eta}.
\end{align*}
\]

Substituting into the equation,

\[
0 = u_{tt} + 3u_{xt} - 4u_{xx} = (16v_{\xi\xi} - 3v_{\xi\eta} + v_{\eta\eta}) + 3(4v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}) - 4(v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta})
\]

\[ = -5v_{\xi\eta}. \]

The general solution of the equation \( v_{\xi\eta} = 0 \) is equal to

\[ v(\xi, \eta) = F(\xi) + G(\eta) \]

for any two twice continuously differentiable functions \( F \) and \( G \). Hence, the solution \( u \) is equal to

\[ u(x, t) = F(x - 4t) + G(x + t) \]
where $F$ and $G$ are as above. The derivative $u_t(x, t)$ is equal to $u_t(x, t) = -4F'(x - 4t) + G'(x + t)$. Hence the initial conditions give

\[
F(x) + G(x) = \phi(x) \\
-4F'(x) + G'(x) = \psi.
\]

Integrating the second equation one gets

\[
-4F(x) + G(x) = \int_0^x \psi(y) \, dy + C
\]

where $C = -4F(0) + G(0)$. So we have a system of equations for $F$ and $G$

\[
F(x) + G(x) = \phi(x) \\
-4F(x) + G(x) = \int_0^x \psi(y) \, dy + C.
\]

Subtracting the second equation from the first and dividing by 5 gives

\[
F(x) = \frac{1}{5} \phi(x) - \frac{1}{5} \int_0^x \psi(y) \, dy - \frac{C}{5}.
\]

Multiplying the first equation by 4, adding to the second and dividing by 5 gives,

\[
G(x) = \frac{1}{5} \phi(x) + \frac{1}{5} \int_0^x \psi(y) \, dy + \frac{C}{5}.
\]

Consequently,

\[
u(x, t) = F(x - 4t) + G(x + t) = \frac{\phi(x - 4t) + \phi(x + t)}{5} + \frac{1}{5} \int_{x - 4t}^{x + t} \psi(y) \, dy.
\]

**Problem 2.** Solve the following Cauchy problem:

\[
\begin{aligned}
&u_{tt} - 4u_{xx} = e^x + \sin t & (x, t) \in \mathbb{R} \times (0, \infty) \\
u(x, 0) = 0 & x \in \mathbb{R} \\
u_t(x, 0) = \frac{1}{1 + x^2} & x \in \mathbb{R}
\end{aligned}
\]

**Solution:** The solution $u$ of is given by

\[
u(x, t) = \frac{\phi(x + 2t) + \phi(x - 2t)}{2} + \frac{1}{4} \int_{x - 2t}^{x + 2t} \psi(y) \, dy + \frac{1}{4} \int_0^t \left( \int_{x - 2(t-s)}^{x+2(t-s)} f(y, s) \, dy \right) ds
\]

where $\phi = 0$, $\psi = \frac{1}{1 + y^2}$, and $f(y, s) = e^y + \sin s$. So,

\[
u(x, t) = \frac{1}{4} \int_{x - 2t}^{x + 2t} \frac{1}{1 + y^2} \, dy + \frac{1}{4} \int_0^t \left( \int_{x - 2(t-s)}^{x+2(t-s)} e^y + \sin s \, dy \right) ds.
\]

The first integral is equal to

\[
\frac{1}{4} \int_{x - 2t}^{x + 2t} \frac{1}{1 + y^2} \, dy = \frac{1}{4} \tan^{-1} y \bigg|_{x - 2t}^{x + 2t} = \tan^{-1}(x + 2t) - \tan^{-1}(x - 2t)
\]
The second integral is equal to
\[
\frac{1}{4} \int_0^t \left( \int_{x-2(t-s)}^{x+2(t-s)} e^y + \sin s \, dy \right) \, ds
\]
\[
= \frac{1}{4} \int_0^t \left( 4(t-s) \sin s \, ds + \left( e^{x+2(t-s)} - e^{x-2(t-s)} \right) \right) \, ds
\]
\[
= t - \sin t + \frac{e^x}{4} \left( e^{2t} + e^{-2t} - e^t - e^{-t} \right).
\]
So the solution is equal to
\[
u(x) = \frac{\tan^{-1}(x + 2t) - \tan^{-1}(x - 2t)}{4} + t - \sin t + \frac{e^x}{4} \left( e^{2t} + e^{-2t} - e^t - e^{-t} \right).
\]

**Problem 3.**

(a): Solve the following initial value problem:
\[
\begin{align*}
  u_{tt} - u_{xx} &= 0 & (x, t) & \in (0, \infty) \times (0, \infty) \\
  u_x(0, t) &= 0 & t & \in (0, \infty) \\
  u(x, 0) &= \phi(x) & x & \in [0, \infty) \\
  u_t(x, 0) &= \psi(x) & x & \in [0, \infty)
\end{align*}
\]

where \( \phi \) and \( \psi \) are \( C^1 \) functions on \([0, \infty)\) satisfying \( \phi'(0) = \psi'(0) = 0 \).

*Hint:* Extend \( \phi \) and \( \psi \) as even functions \( \tilde{\phi} \) and \( \tilde{\psi} \) on \( \mathbb{R} \). Solve the Cauchy problem with \( \tilde{\phi} \) and \( \tilde{\psi} \) as initial data and show that the restriction the solution to \((0, \infty) \times (0, \infty)\) is a solution of the above problem.

(b): Solve the problem with \( \phi(x) = x^3 + x^6 \) and \( \psi(x) = \sin^3 x \).

**Solution:** Define
\[
\tilde{\phi}(x) = \begin{cases} 
  \phi(x) & x \geq 0 \\
  \phi(-x) & x \leq 0
\end{cases}
\quad \text{and} \quad
\tilde{\psi}(x) = \begin{cases} 
  \psi(x) & x \geq 0 \\
  \psi(-x) & x \leq 0
\end{cases}
\]

Then \( \phi \) and \( \psi \) are even functions on \( \mathbb{R} \). Let \( \tilde{u} \) be the solution of
\[
\begin{align*}
  \tilde{u}_{tt} - \tilde{u}_{xx} &= 0 & (x, t) & \in \mathbb{R} \times (0, \infty) \\
  \tilde{u}(x, 0) &= \tilde{\phi}(x) & x & \in \mathbb{R} \\
  \tilde{u}_t(x, 0) &= \tilde{\psi}(x) & x & \in \mathbb{R}.
\end{align*}
\]

Set \( u(x, t) = \tilde{u}(x, t) \). Since \( \tilde{u} \) solves the equation at points \((x, t)\) with \( x > 0 \), the function \( u \) satisfies \( u_{tt} - u_{xx} = 0 \) on \( \mathbb{R} \times (0, \infty) \). Moreover, for \( x > 0 \), \( u(x, 0) = \tilde{u}(x, 0) = \tilde{\phi}(x) = \phi(x) \) (by definition of \( \tilde{\phi} \) and \( u_t(x, 0) = \tilde{u}_t(x, 0) = \tilde{\psi}(x) = \psi(x) \) (by definition of \( \tilde{\psi} \)). It remains to show that \( u_t(0, t) = 0 \). It suffices to show that \( \tilde{u}(x, t) \) is even with respect to \( x \) since the derivative of an even function is odd so that \( 0 = \tilde{u}_t(x, 0) = u_t(x, 0) \). To see this set \( v(x, t) = \tilde{u}(-x, t) \). Then
\[
v_{tt} - v_{xx} = u_{tt} - \tilde{u}_t - \tilde{u}_{xx} = 0
\]
and \( v(x, 0) = \tilde{u}(-x, 0) = \tilde{\phi}(-x) = \phi(x) \) and \( v_t(x, 0) = \tilde{u}_t(-x, 0) = \tilde{\psi}(-x) = \psi(x) \). So \( v \) is a solution of the initial value problem and since \( \tilde{u} \) is another solution of the initial value problem, we have that \( v = \tilde{u} \), that is \( \tilde{u}(-x, t) = \tilde{u}(x, t) \).
(b) With the notation as above the solution \( u \) of the initial value problem is equal to

\[
u(x,t) = \frac{1}{2}[\tilde{\phi}(x+t) + \tilde{\phi}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{\psi}(y) \, dy.
\]

To express this solution in terms of \( \phi \) and \( \psi \), one needs to consider to cases.

**Case 1** The point \((x,t) \in (0, \infty) \times (0, \infty)\) satisfies \(x-t \geq 0\). Then \(\tilde{\phi}(x+t) = \phi(x+t)\), \(\tilde{\phi}(x-t) = \phi(x-t)\) and \(\tilde{\psi}(y) = \psi(y)\) for \(y \geq 0\). So, in this case,

\[
u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(y) \, dy.
\]

**Case 2** The point \((x,t) \in (0, \infty) \times (0, \infty)\) satisfies \(x-t < 0\). In this case,

\[
u(x,t) = \frac{1}{2}[\tilde{\phi}(x+t) + \tilde{\phi}(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{\psi}(y) \, dy
\]

\[
= \frac{1}{2}[\tilde{\phi}(x+t) + \tilde{\phi}(t-x)] + \frac{1}{2} \int_{x-t}^{0} \tilde{\psi}(y) \, dy + \frac{1}{2} \int_{0}^{x+t} \tilde{\psi}(y) \, dy
\]

\[
= \frac{1}{2}[\phi(x+t) + \phi(t-x)] + \frac{1}{2} \int_{0}^{t-x} \psi(y) \, dy + \frac{1}{2} \int_{0}^{x+t} \psi(y) \, dy
\]

where we have used that \(\tilde{\phi}\) and \(\tilde{\psi}\) are even and that on \((0, \infty)\) they are equal to \(\phi\) and \(\psi\), respectively.