MATH 412 Fourier Series and PDE–Spring 2010

HOMEWORK 5– Due: Thursday, April 8.

Submit solutions to all of the problems. Each problem is worth the same number of points. Collaboration is allowed, but you need to turn in individually written solutions. The problem are from the book “An introduction to partial differential equations” by Y. Pinchover and J. Rubinstein.

Problem 1.

(a) Solve the following Sturm-Liouville problem

\[
\begin{aligned}
(xu')' + \frac{\lambda}{x} u &= 0, & 1 < x < c \\
u(1) &= u'(c) = 0.
\end{aligned}
\]

(b) Show directly that the eigenfunctions are orthogonal with respect to the suitable inner product. What is the length of an eigenfunction with respect to the norm related to this inner product?

Problem 2.

(a) Consider the following Sturm-Liouville problem

\[
\begin{aligned}
(x^2v')' + \lambda v &= 0, & 1 < x < b \\
v(1) &= u(b) = 0
\end{aligned}
\]

where \( b > 1 \). Find the eigenvalues and eigenfunctions of the problem.

(b) Show directly that the eigenfunctions are orthogonal with respect to the suitable inner product. What is the length of an eigenfunction with respect to the norm related to this inner product?

Problem 3. Use the method of separation of variables to solve the telegraphic equation with the initial boundary conditions

\[
\begin{aligned}
u_{tt} + u_t - u_{xx} &= 0, & 0 < x < 2, t > 0 \\
u(0, t) &= u(2, t) = 0, & t \geq 0 \\
u(x, 0) &= 0, & 0 \leq x \leq 2 \\
u_t(x, 0) &= x, & 0 \leq x \leq 2.
\end{aligned}
\]

Problem 4. Use the method of separation of variables to solve

\[
\begin{aligned}
u_t &= u_{xx} - 4u, & 0 < x < \pi, t > 0 \\
u_x(0, t) &= u(\pi, t) = 0, & t \geq 0 \\
u(x, 0) &= x^2 - \pi^2, & 0 \leq x \leq \pi
\end{aligned}
\]

Is the solution found above a classical solution?