4 Metric Spaces

Let $X$ be a nonempty set. A distance function or a metric on a set $X$ is a function which assigns to a pair of points $x$ and $y$ in $X$ a real number $d(x, y)$ having the following properties:

- **M1** $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- **M2** $d(x, y) = d(y, x)$ for all $x, y \in X$.
- **M3** (triangle inequality): $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y$ and $z \in X$.

**Definition 4.1.** A metric space is a pair $(X, d)$ where $d$ is a metric defined on the set $X$.

Given $a \in X$ and $r > 0$, the sets

$$B(a, r) = B_X(a, r) = \{x \in X \mid d(a, x) < r\}$$
$$\overline{B}(a, r) = \overline{B}_X(a, r) = \{x \in X \mid d(a, x) \leq r\}$$

are called the open ball and the closed ball with center at $a$ and radius $r$.

**Example 4.2.** The set of real numbers $\mathbb{R}$ is a metric space with the standard metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$. The open ball $B_r(a)$ with center at $a$ and radius $r > 0$ is equal to the open interval $(a - r, a + r)$.

**Example 4.3.** Let $X$ be any nonempty set and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} 0 & x = y, \\ 1 & x \neq y. \end{cases}$$

Then $d$ is a metric called the discrete metric on $X$.

**Example 4.4.** Let $(X, d)$ be a metric space and $Y$ is a nonempty subset of $X$. Then the restriction of $d$ to $Y \times Y$, $d_Y = d|Y \times Y$, is a metric on $Y$. It is called the induced metric. The pair $(Y, d_Y)$ is a metric space, metric subspace of $X$. (We will refer to $Y$ as a subspace of $X$, rather than $(Y, d_Y)$ as a subspace of $(X, d)$.) The open ball and closed balls in $(Y, d_Y)$ are

$$B_Y(a, r) = B_X(a, r) \cap Y \quad \text{and} \quad \overline{B}_Y(a, r) = \overline{B}_X(a, r) \cap Y.$$
Example 4.5. Let $(X_i, d_i), 1 \leq j \leq n,$ be metric spaces and let $X = X_1 \times \ldots \times X_n$. Then each of the following functions defines a metric (product metric) on $X$:

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$$
$$d(x, y) = \left[ \sum_{i=1}^{n} d_i(x_i, y_i)^2 \right]^{1/2}$$
$$d(x, y) = \max_{1 \leq j \leq n} d_i(x_i, y_j)$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ in $X$. The pair $(X, d)$ defined above is called the metric product or just a product) of $(X_i, d_i)$. The open ball and closed balls in $X$ are

$$B_X(a, r) = B_{X_1}(a_1, r) \times \ldots \times B_{X_n}(a_n, r)$$
$$\overline{B}_X(a, r) = B_{X_1}(a_1, r) \times \ldots \times \overline{B}_{X_n}(a_n, r)$$

where $a = (a_1, \ldots, a_n) \in X$.

A subset $U$ of a metric space $X$ is called a neighborhood of $a \in X$, if there is $r > 0$ such that $B(a, r) \subseteq U$.

Example 4.6. Clearly, $B(a, r)$ and $\overline{B}(a, r)$ are neighborhoods of $a$. If $U_1, U_2$ are neighborhoods of $a$, then their union and intersection are neighborhoods of $a$. Indeed, let $B(a, r_j) \subseteq U_j$ for $j = 1, 2$. Then $B(a, r) \subseteq U_1 \cap U_2 \subseteq U_1 \cup U_2$ where $r = \min\{r_1, r_2\}$.

4.1 Norms and normed vector spaces

We next define the class of metric spaces which are the most interesting in analysis. Let $X$ be a vector space over $\mathbb{R}$. A norm is a function $\| \cdot \| : X \to [0, \infty)$ having the following properties.

N1 $\| x \| = 0$ if and only if $x = 0$.
N2 $\| \alpha x \| = |\alpha| \| x \|$ for all $x \in X$ and $\alpha \in \mathbb{R}$.
N3 $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$.

A pair $(X, \| \cdot \|)$ is called a normed vector space.
Proposition 4.7. Let \((X, \|\cdot\|)\) be a normed space. Then the function
\[
d : X \times X \to \mathbb{R}, \quad d(x, y) = \|x - y\|
\]
is a metric on \(X\). It is called the metric induced from the norm.

Proof. The axioms M1 and M2 are clear. If \(x, y\) and \(z \in X\), then, in view of N3,
\[
d(x, z) = \|x - z\| = \|(x - y) + (y - z)\|
\]
\[
\leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),
\]
and so the triangle inequality follows. \(\square\)

Example 4.8. The absolute value \(|\cdot|\) on \(\mathbb{R}\) is a norm.

Example 4.9 (Induced norm). Let \(Y\) be a nonempty subspace of a normed space \((X, \|\cdot\|)\). Then the restriction \(\|\cdot\|_Y = \|\cdot\||Y\) is a norm on \(Y\). It is called the induced norm on \(Y\).

Example 4.10 (Product norm). Let \((X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)\) be normed spaces over \(\mathbb{R}\) and let \(X = X_1 \times \ldots \times X_n\). Then each of the following function defines a norm on \(X\):
\[
\|x\| := \sum_{i=1}^{n} \|x_i\|_i
\]
\[
\|x\| := \left[\sum_{i=1}^{n} \|x_i\|_i^2\right]^{1/2}
\]
\[
\|x\| = \max\{\|x_1\|_1, \ldots, \|x_n\|_n\}
\]
where \(x = (x_1, \ldots, x_n) \in X\). Either of them is called the product norm on \(X\).

Example 4.11 (Space of Bounded Functions). Let \(X\) be a nonempty set and let \((Y, \|\cdot\|)\) be a normed space over \(\mathbb{R}\). A function \(f : X \to Y\) is called bounded if its image is bounded in \(Y\), i.e., \(f(X) \subset B_Y(0, r)\) for some \(r > 0\). Explicitly, this means that there is \(r > 0\) so that \(\|f(x)\| < r\) for all \(x \in X\). The set of all bounded function \(f : X \to Y\) is denoted by \(B(X, Y)\).

This is a vector space, called space of bounded functions from \(X\) to \(Y\). If \(f \in B(X, y)\), define
\[
\|f\|_\infty = \sup\{\|f(x)\| \mid x \in X\}.
\]
If \( f, g \in B(X,Y) \) and \( x \in X \), then
\[
\|f(x) + g(x)\| \leq \|f(x)\| + \|g(x)\| \leq \|f\|_\infty + \|g\|_\infty
\]
and since \( x \) was arbitrary element in \( X \) one concludes that
\[
\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.
\]
Hence \( \|\cdot\|_\infty \) satisfies the triangle inequality. The other axioms are also satisfied. So, \((B(X,Y),\|\cdot\|_\infty)\) is a normed space.

### 4.2 Inner product spaces

Let \( X \) be a vector space over \( \mathbb{R} \). Then a function \( \langle \cdot, \cdot \rangle : X \times X \to K \) is called the inner product or scalar product, if it satisfies the following conditions.

**IP1** \( \langle x, x \rangle \geq 0 \) and \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).

**IP2** \( \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \), for all \( x, y, z \in X \) and all \( \alpha, \beta \in \mathbb{R} \).

**IP3** \( \langle x, y \rangle = \langle y, x \rangle \) for all \( x, y \in X \).

A pair \((X, \langle \cdot, \cdot \rangle)\) is called an inner product space.

**Example 4.12.** Consider \( X = \mathbb{R}^n \) and for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \) let
\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.
\]
Then \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathbb{R}^n \). It is called the standard inner product.

**Proposition 4.13 (Cauchy-Schwarz Inequality).** Let \((X, \langle \cdot, \cdot \rangle)\) be an inner product space. Then
\[
|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}
\]
for all \( x, y \in X \) with the equality if and only if \( x \) and \( y \) are linearly dependent.

**Proof.** Fix two points \( x, y \in X \). Without loss of generality we may assume that \( y \neq 0 \) (if \( y = 0 \) then the claim follows since both sides are equal to 0). For every \( t \in \mathbb{R} \), we have
\[
0 \leq \langle x - ty, x - ty \rangle = \langle x, x \rangle - 2t \langle x, y \rangle + t^2 \langle y, y \rangle.
\]
The right-hand side defines a quadratic function \( f(t) = \langle x, x \rangle - 2t \langle x, y \rangle + t^2 \langle y, y \rangle \). The function \( f \) has a minimum at \( t_0 = \frac{\langle x, y \rangle}{\langle y, y \rangle} \). Evaluating \( f(t_0) \) we get
\[
0 \leq f(t_0) = \langle x, x \rangle - 2 \left( \frac{\langle x, y \rangle}{\langle y, y \rangle} \right) \langle x, y \rangle + \left( \frac{\langle x, y \rangle}{\langle y, y \rangle} \right)^2 \langle y, y \rangle = \langle x, x \rangle - \frac{\langle x, y \rangle^2}{\langle y, y \rangle}
\]
implying that \( \langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle \), that is \( |\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \).

Corollary 4.14 (Classical Cauchy-Schwarz inequality). If \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), then
\[
\left| \sum_{j=1}^n x_j y_j \right|^2 \leq \left( \sum_{j=1}^n |x_j|^2 \right) \left( \sum_{j=1}^n |y_j|^2 \right).
\]

Proposition 4.15. Let \((X, \langle \cdot, \cdot \rangle)\) be an inner product space and let
\[
\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in X.
\]
Then \( \| \cdot \| \) is a norm on \( X \), the norm induced from the inner product.

Proof. It suffices to prove the triangle inequality. Let \( x, y \) and \( z \in X \). For \( x, y \in X \), we have
\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + 2 \langle x, y \rangle + \langle y, y \rangle \leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2
\]
So, \( \|x + y\| \leq \|x\| + \|y\| \) as claimed. \( \blacksquare \)