10 Compactness in function spaces: Ascoli-Arzelá theorem

Recall that if \((X, d)\) is a compact metric space then the space \(C(X)\) is a vector space consisting of all continuous function \(f : X \to \mathbb{R}\). The space \(C(X)\) is equipped with the norm \(\|f\| := \max\{|f(x)| \mid x \in X\}\). The norm induces the metric

\[
\sigma(f, g) = \|f - g\| = \max\{|f(x) - g(x)| \mid x \in X\}.
\]

**Definition 10.1.** The family \(F \subset C(X)\) is called **equicontinuous** if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that

\[
|f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in X \text{ satisfying } d(x, y) < \delta \text{ and all } f \in F.
\]

The family \(F\) is called **equibounded** if there is a constant \(M\) such that

\[
|f(x)| \leq M
\]

for all \(f \in F\) and all \(x \in X\).

**Example 10.2.** Let \((X, d)\) be a metric space and let \(M > 0\). By \(F\) denote the family of functions \(f : X \to \mathbb{R}\) satisfying

\[
|f(x) - f(y)| \leq Md(x, y) \quad \text{for all } x, y \in X.
\]

Then the family \(F\) is equicontinuous. To see this, take \(\varepsilon > 0\) and set \(\delta = \varepsilon/M\). Then for every \(y \in B_\delta(x)\) and every \(f \in F\), we have

\[
|f(x) - f(y)| \leq Md(x, y) < M \cdot \varepsilon/M = \varepsilon.
\]

Let \(M > 0\) and let \(F\) be the set all differentiable functions \(f : [a, b] \to \mathbb{R}\) satisfying \(|f'(x)| \leq M\) for all \(x \in (a, b)\). In view of the mean value theorem,

\[
|f(x) - f(y)| \leq M |x - y|
\]

for all \(x, y \in (a, b)\). Using the previous example, \(F\) is a equicontinuous family.

If \((X, d)\) is a metric space, we call a subset \(F \subset X\) **precompact** (or **relatively compact**) if \(\overline{F}\) is compact in \(X\).

**Theorem 10.3 (Ascoli–Arzelá Theorem).** Let \((X, d)\) be a compact space. A subset \(F\) of \(C(X)\) is relatively compact if and only if \(F\) is equibounded and equicontinuous.

**Proof.** Assume that \(F\) is relatively compact. This is means that \(\overline{F}\) is compact. We claim that \(F\) is equibounded and equicontinuous. Since \(\overline{F}\) compact, it is totally bounded. In particular, \(F\) is totally bounded. This implies that \(F\) is equibounded. To see that \(F\) is equicontinuous, take \(\varepsilon > 0\). Then, there are \(f_1, \ldots, f_N \in C(X)\) such that

\[
\forall \varepsilon/3, \exists f_1, \ldots, f_N \in C(X) \text{ such that } \forall \varepsilon > 0, \exists f_1, \ldots, f_N \in C(X) \text{ such that } F \subset B_{\varepsilon/3}(f_1) \cup \ldots \cup B_{\varepsilon/3}(f_N).
\]
Each $f_i$ is uniformly continuous since $(X, d)$ is compact. Hence there exists $\delta > 0$ such that
\[ |f_i(x) - f_i(y)| < \varepsilon/3 \quad \text{for all } x, y \text{ such that } d(x, y) < \delta \text{ and } 1 \leq i \leq N. \]
Given $f \in \mathcal{F}$, in view of (10) there is $1 \leq j \leq N$ such that
\[ \sigma(f, f_j) < \varepsilon/3. \]
Now if $x, y \in X$ satisfy $d(x, y) < \delta$, then
\[
|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\
\leq \sigma(f, f_j)|f_j(x) - f_j(y)| + \sigma(f, f_j) < 3\varepsilon/3 = \varepsilon
\]
showing that $\mathcal{F}$ is equicontinuous.
Conversely, assume that $\mathcal{F}$ is equibounded and equicontinuous. It suffices to show that $\mathcal{F}$ is totally bounded. Indeed, if $\mathcal{F}$ is totally bounded, then $\mathcal{F}$ is totally bounded and since $C(X)$ is complete, the set $\mathcal{F}$ is also complete. Hence $\mathcal{F}$ is compact. Take $\varepsilon > 0$. Since $\mathcal{F}$ is equicontinuous, for every $x \in X$ there exists $\delta_x > 0$ such that
\[ |f(x) - f(y)| < \varepsilon/4 \quad \text{for all } y \text{ such that } d(x, y) < \delta_x \text{ and } f \in \mathcal{F}. \]
The collection $\{B_{\delta_x}(x)\}_{x \in X}$ is an open cover of a compact metric space $X$. Hence there are $x_1, \ldots, x_N$ such that
\[ X = B_{\delta_{x_1}} \cup \ldots \cup B_{\delta_{x_N}}. \]
In particular,
\[ |f(x) - f(x_i)| < \varepsilon/4 \quad \text{for all } x \in B_{\delta_i}(x_i) \text{ and } f \in \mathcal{F}, \]
where we have abbreviated $\delta_i = \delta_{x_i}$. Since $\mathcal{F}$ is equibounded, the set $F := \{f(x_i) | 1 \leq i \leq N, f \in \mathcal{F}\}$ is bounded. Since a bounded set in $\mathbb{R}$ (with the standard metric) is totally bounded, there are points $y_1, \ldots, y_K$ in $\mathbb{R}$ such that
\[ F \subset \bigcup_{1 \leq i \leq K} B_{\varepsilon/4}(y_i). \]
For any map $\varphi : \{1, \ldots, N\} \to \{1, \ldots, K\}$, define
\[ \mathcal{F}_\varphi := \{f \in \mathcal{F} | f(x_i) \in B_{\varepsilon/4}(y_{\varphi(i)}), i = 1, \ldots, N\}. \]
Note that there are only finitely many sets $\mathcal{F}_\varphi$ and that every $f \in \mathcal{F}$ belongs to one of the sets $\mathcal{F}_\varphi$. We claim that the diameter of $\mathcal{F}_\varphi$ is finite. Indeed, take $f, g \in \mathcal{F}_\varphi$ and $x \in X$. Then $x \in B_{\delta_i}(x_i)$ for some $i$ and
\[
|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - y_{\varphi(i)}| + |y_{\varphi(i)} - g(x_i)| + |g(x_i) - g(x)| \leq 4 \cdot \varepsilon/4 = \varepsilon.
\]
Hence $\sigma(f, g) < \varepsilon$ showing that $\text{diam } \mathcal{F}_\varphi \leq \varepsilon$. Consequently, $\mathcal{F}$ can be covered by finitely many sets of diameter less than $\varepsilon$. Hence $\mathcal{F}$ is totally bounded and the proof is complete. ■
A simple consequence we have the following corollary.

**Corollary 10.4.** Let $X$ be a compact metric space and let $(f_n) \subset C(X)$ be a sequence which is equibounded and equicontinuous in $C(X)$. Then every the sequence $(f_n)$ has a uniformly convergent subsequence.

### 11 Structure of complete metric spaces-Baire’s theorem

Let $(X,d)$ be a metric space. A subset $U$ of $X$ is called dense if $\overline{U} = X$. If $U$ and $V$ are open and dense, then $U \cap V$ is also open and dense. To see that $U \cap V$ is dense, we have to show that $O \cap U \cap V$ is non-empty for any open set $O$. Since $U$ is dense, there is $u \in O \cap U$, and since $O \cap U$ is open, $B(u,r) \subset O \cap U$ for some $r > 0$. Since $V$ is dense, $B(u,r) \cap V \neq \emptyset$ so that, $\emptyset \neq B(u,r) \cap V \subset O \cap U \cap V$. If $U$ and $V$ are assumed to be dense but not necessarily open, then the intersection $U \cap V$ does not have to be dense. For example, let $U$ be the set of rational numbers and $V$ the set of irrational numbers $\mathbb{Q}^c$. Then both sets are dense in $\mathbb{R}$ with the usual metric, however, $U \cap V = \emptyset$. Consider, now a sequence of dense and open sets $U_n$. In general, the intersection $\bigcap_{n \geq 1} U_n$ may be empty.

**Example 11.1.** Consider $(\mathbb{Q},d)$ with the usual metric $d$. Let $\{q_n|n \in \mathbb{N}\}$ be an enumeration of rational numbers, and let $U_n = \mathbb{Q} \setminus \{q_n\}$. Then each $U_n$ is open since it is a complement of a closed set $\{q_n\}$, and is dense. However, $\bigcap_{n \geq 1} U_n = \bigcap_{n \geq 1} (\mathbb{Q} \setminus \{q_n\}) = \mathbb{Q} \setminus \bigcup_{n \geq 1} \{q_n\} = \emptyset$.

A subset $F$ of $X$ is called nowhere dense if $(F)^o = \emptyset$.

**Theorem 11.2 (Baire).** Let $(X,d)$ be a complete metric space. Then:

(a) If $\{U_n\}$ is a sequence of open and dense subsets of $X$, then $\bigcap_{n \geq 1} U_n$ is dense.

(b) If $\{F_n\}$ is a sequence of nowhere dense subsets of $X$, then $\bigcup F_n$ has empty interior.

**Proof.** (a) It suffices to show that $B(x,r)$ contains a point belonging to $\bigcap_{n \geq 1} U_n$ for any open ball $B(x,r)$. Since $U_1$ is open and dense, $B(x,r) \cap U_1$ is non-empty and open. So, there exists an open ball $B(x_1,R)$ with $R < 1$ such that $B(x_1,R) \subseteq B(x,r)$ and $B(x_1,R) \subseteq U_1$. Taking $r_1 < R$, we get that $B(x_1,r_1) \subseteq B(x,r)$ and $B(x_1,r_1) \subseteq U_1$. Similarly, since $U_2$ is open and dense, there exists $x_2$ and $r_2 < 1/2$ such that $B(x_2,r_2) \subseteq B(x_1,r_1) \cap U_2$. Continuing in this way we find a sequence of balls $B(x_n,r_n)$ with $r_n < 1/n$ and $B(x_{n+1},r_{n+1}) \subseteq B(x_n,r_n) \cap U_n$. We claim that $\{x_n\}$ is Cauchy. By construction, $B_n(x_n,r_n) \subseteq B(x_n,r_n) \cap U_n$. We claim that $\{x_n\}$ is Cauchy. By construction, $B_n(x_n,r_n) \subseteq B(x_n,r_n)$ for all $n \geq k$. Given $\varepsilon > 0$ choose $k \in \mathbb{N}$ so that $1/k < \varepsilon/2$. Then, if $n,m \geq k$,

$$d(x_n,x_m) \leq d(x_n,x_k) + d(x_k,x_m) < 1/k + 1/k < \varepsilon.$$
Because \((X, d)\) is complete, \(\{x_n\}\) converges, say to \(y\). The point \(y\) lies in all balls \(\overline{B}(x_k, r_k)\) since \(x_n \in \overline{B}(x_k, r_k)\) for all \(n \geq k\) and \(\overline{B}(x_k, r_k)\) is closed for all \(k\), so that after taking a limit as \(n \to \infty\), \(y \in \overline{B}(x_k, r_k)\) for all \(k\). In particular, \(y \in \overline{B}(x_1, r_1) \subseteq B(x, r)\) and \(y \in \overline{B}(x_{n+1}, r_{n+1}) \subset U_n\) for all \(n\). Consequently, \(y \in B(x, r) \cap \bigcap_{n \geq 1} U_n\), and the proof is finished.

(b) Arguing by contradiction assume that \(\bigcup F_n\) has non-empty interior. So \(B(x, r) \subseteq \bigcup F_n\) for some \(x\) and \(r > 0\). Define \(U_n = X \setminus \overline{F_n}\). Clearly, \(U_n\) is open and we claim that it is dense. Indeed, if for some open set \(V\) we have \(V \cap U_n = \emptyset\), then \(V \subset X \setminus U_n = \overline{F_n}\) contradicting that \(\overline{F_n}\) has empty interior. By (a), \(\bigcap_{n \geq 1} U_n\) is dense. So \(B(x, r) \cap \bigcap_{n \geq 1} U_n \neq \emptyset\). On the other hand, \(B(x, r) \subseteq \bigcup F_n \subseteq \bigcup \overline{F_n}\), so that \(\emptyset = B(x, r) \cap \bigcup \overline{F_n} = B(x, r) \cap \bigcap_{n \geq 1} [X \setminus \overline{F_n}] = B(x, r) \cap \bigcap_{n \geq 1} U_n\), a contradiction. \(\blacksquare\)

**Example 11.3.** The metric space \(\mathbb{R}\) with the standard metric cannot be written as a countable union of nowhere dense sets since it is complete. By contrast, \(\mathbb{Q}\) with the standard metric can be written as the union of one point sets \(\{q_n\}\), where \(\{q_n\} n \in \mathbb{N}\) is an enumeration of \(\mathbb{Q}\). Every one point set \(\{q_n\}\) is closed in \(\mathbb{Q}\) and its interior is empty, so nowhere dense. This does not contradict Baire’s theorem since \(\mathbb{Q}\) with the standard metric is not complete.

**Application of the Baire’s theorem**

**Theorem 11.4.** There exists a continuous function \(f : [0, 1] \to \mathbb{R}\) which is not differentiable at any point \(x \in [0, 1]\).

**Proof.** Recall that \(f\) has a right-hand derivative at \(x\) if

\[
\lim_{h \to 0^+} \frac{[(f(x + h) - f(x))/h]} \text{ exists.}
\]

We denote this limit by \(f'_+(x)\). In particular, if \(f\) is differentiable at \(x \in [0, 1]\) then \(f'_+(x)\) exists and equals \(f'(x)\). Let

\[D = \{f \in C([0, 1], \mathbb{R}) \text{ such that } f'_+(x) \text{ exists}\}\]

and let \(D_{n,m}\) be the set of all \(f \in C([0, 1], \mathbb{R})\) for which there exists some \(x \in [0, 1 - 1/m]\) such that

\[|f(x + h) - f(x)| \leq n \cdot h \text{ for all } h \in [0, 1/m].\]

We shall show that \(D \subseteq \bigcup_{n,m} D_{n,m}\) and that each \(D_{n,m}\) is nowhere dense. By Theorem 11.2 (b), \(\bigcup_{n,m} D_{n,m}\) has empty interior. Consequently, the set \(D\) has empty interior showing that \(D \neq C[0, 1]\).

**Claim 1:** \(D \subseteq \bigcup_{n,m} D_{n,m}\). To see this, let \(f \in D\). Then there exists \(x \in [0, 1]\) such that \(f'_+(x)\) exists. In particular,

\[
\lim_{h \to 0^+} \left| \frac{f(x + h) - f(x)}{h} \right| = |f'_+(x)|. \tag{1}
\]
Take an integer $n$ such that $|f'_+(x)| < n$. Then there exists $\delta > 0$ such that

$$|f(x + h) - f(x)| \leq n \cdot h \quad \text{for all } 0 \leq h \leq \delta.$$ 

Now choose $m \in \mathbb{N}$ large so that $x \leq 1 - 1/m$ and $1/m < \delta$. Then $f \in D_{n,m}$ as claimed.

**Claim 2:** $D_{n,m}$ is closed. Take a sequence $(f_k) \subset D_{n,m}$ such that $\sigma(f_k, f) \to 0$ for some $f \in C[0,1]$. To prove the claim we have to show that $f \in D_{n,m}$. Since $f_k \in D_{n,m}$, there is a sequence $(x_k)$ satisfying $0 \leq x_k \leq 1 - 1/m$ and

$$|f_k(x_k + h) - f_k(x_k)| \leq nh \quad \text{for all } 0 \leq h \leq 1 - 1/m. \quad (2)$$

Without loss of generality we may assume that $x_k \to x_\ast \in [0,1-1/m]$. Then, using the triangle inequality,

$$|f(x_\ast + h) - f(x_\ast)| \leq |f(x_\ast + h) - f(x_k + h)| + |f(x_k + h) - f_k(x_k + h)|$$
$$+ |f_k(x_k + h) - f_k(x_k)| + |f_k(x_k) - f_k(x_\ast)|$$
$$+ |f_k(x_\ast) - f(x_\ast)|$$
$$\leq |f(x_\ast + h) - f(x_k + h)| + |f_k(x_k) - f_k(x_\ast)|$$
$$+ 2d(f_k, f) + n \cdot h$$

for all $0 \leq h \leq 1/m$. Since $d(f_k, f) \to 0$, $|f(x + h) - f(x_k + h)| \to 0$, and $|f_k(x_k) - f(x_k)| \to 0$, we conclude that

$$|f(x + h) - f(x)| \leq n \cdot h$$

for all $0 \leq h \leq 1/m$. Consequently, $f \in D_{n,m}$ showing that $D_{n,m}$ is closed.

**Claim 3:** $D_{n,m}^\circ = \emptyset$. This together with Claim 2 implies that $M_m$ is nowhere dense. To prove the claim it suffices to show that if $f \in D_{n,m}$, then any open ball $B_\varepsilon$ contains $g$ which doesn’t belong to $D_{n,m}$. Take a piecewise linear function $g : [0,1] \to \mathbb{R}$ such that $\sigma(g, f) = \sup\{|f(x) - g(x)| \mid 0 \leq x \leq 1\} < \varepsilon$ and $|g'_+(x)| > n$ for all $x \in [0,1)$. Then $g \in B_\varepsilon(f)$ and $g \notin D_{n,m}$. So, $D_{n,m}^\circ = \emptyset$, as claimed.