On the final exam you will be asked to solve problems using definitions and theorems discussed in class. You will also be asked to give proofs of two of theorems listed below.

1. **Proposition 7.12** A set \( F \subset X \) is closed if and only if every convergent sequence \((x_n)\) such that \( x_n \in F \) converges to a point in \( F \).

2. **Theorem 7.19** Let \( f : X \to Y \) be a function from a metric space \((X, d)\) to \((Y, \rho)\). Then the following are equivalent.
   
   (i) \( f \) is continuous.
   
   (ii) \( f^{-1}(U) \) is open in \( X \) for every open subset \( U \) of \( Y \).
   
   (iii) \( f^{-1}(F) \) is closed in \( X \) for every closed subset \( F \) of \( Y \).

3. **Theorem 8.10 (Banach Fixed Point Theorem)** Let \((X, d)\) be a complete metric space and \( f : X \to X \) a contraction. Then there exists exactly one point \( u \in X \) such that \( f(u) = u \). Moreover, for every \( x \in X \), the sequence \((f^n(x))\) converges to \( u \).

4. **Theorem 9.3 and Corollary 9.4** Let \((X, d)\) and \((Y, \rho)\) be metric spaces and let \( f : X \to Y \) be continuous. If a subset \( K \subseteq X \) is compact, then \( f(K) \) is compact in \((Y, \rho)\). In particular, if \((X, d)\) is compact, then \( f(X) \) is compact in \( Y \).

   **Extreme value theorem** Let \((X, d)\) be a compact metric space and \( f : X \to \mathbb{R} \) be a continuous function. Then \( f \) attains a maximum and a minimum value, that is, there exist \( a \) and \( b \in X \) such that \( f(a) = \inf \{ f(x) \mid x \in X \} \) and \( f(b) = \sup \{ f(x) \mid x \in X \} \).

5. **Theorem 9.6** Assume that \( f : (X, d) \to (Y, \rho) \) is a continuous mapping and \((X, d)\) is a compact metric space. Then \( f \) is uniformly continuous.

Below is the list of problems that could be helpful in your preparation to the exam.

1. Let \( A \) and \( B \) be nonempty bounded subsets of \( \mathbb{R} \). Show:
   
   (a) Let \( A \) and \( B \) be nonempty bounded subsets of \( \mathbb{R} \). Show that \( \sup(A \cap B) \leq \min\{\sup A, \sup B\} \). Can this inequality be strict?
   
   (b) \( \sup A \) and \( \inf A \) belong to \( \partial A \).

2. Let \( \mathcal{S} \) be the set of all infinite sequences of integers having the property that all but finitely many of the terms are equal to 0. Show that \( \mathcal{S} \) is countable.
3. Let \( C \) be the set of all Cauchy sequences in \( \mathbb{R} \) equipped with the standard metric. If \((x_n), (y_n) \in C\), define \( \sim \) by \((x_n) \sim (y_n)\) if \( |x_n - y_n| \to 0\). Show that \( \sim \) is an equivalence relation.

4. Consider the set \( X = [-1, 1] \) as a subspace of \( \mathbb{R} \) equipped with the standard metric. Which of the following sets are open in \( X \)? Which are open in \( \mathbb{R} \)? Which are closed in \( X \) and which are closed in \( \mathbb{R} \)?

\[
A = \{x \mid 1/2 < |x| < 2\} \quad B = \{x \mid 1/2 < |x| \leq 2\} \\
C = \{x \mid 1/2 \leq |x| \leq 1\} \quad D = \{x \mid 0 < |x| \leq 1 \text{ and } 1/x \notin \mathbb{Z}\}
\]

5. Determine the interior, the closure, and the boundary of each of the following subsets of \( \mathbb{R}^2 \) equipped with the Euclidean metric.

\[
A = \{(x, y) \mid x > 0, \ y \neq 0\} \quad B = \{(x, y) \mid n \in \mathbb{Z}, \ y \in \mathbb{R}\} \\
C = \{(x, y) \mid x \in \mathbb{Q}, \ y \in \mathbb{R}\} \quad D = \{(x, y) \mid 0 < x^2 - y^2 \leq 1\} \\
E = \{(x, y) \mid x \neq 0, \ y \leq 1/x\} \quad F = \{(x, y) \mid x \in \mathbb{Q} \cap (0, 1), \ y \in (0, 1) \setminus \mathbb{Q}\}
\]

6. Let \((X, d)\) be a metric space and \( A \subset X \). Prove or provide a counterexample:

(a) Do \( A \) and \( \overline{A} \) always have the same interior?

(b) Do \( A \) and \( A^\circ \) always have the same closure?

(c) Is it true that \( (A^\circ)^c = \overline{A}^c \)?

7. Let \((X, d)\) be a metric space and \( A, B \subset X \). Show

(a) If \( A \subset B \), then \( \overline{A} \subset \overline{B} \).

(b) \( \overline{A \cup B} = \overline{A} \cup \overline{B} \) and \( \overline{A \cap B} \subset \overline{A} \cap \overline{B} \). Give an example showing that the last inclusion can be proper.

(c) True or false? \( (A \cup B)^\circ = A^\circ \cup B^\circ \).

8. Let \( f : (X, d) \to (Y, \rho) \). Show that \( f \) is continuous if and only if \( \partial f^{-1}(A) \subset f^{-1}(\partial A) \) for every subset \( A \subset Y \).

9. Show that if \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) are uniformly continuous, then \( f + g \) and \( \alpha f \) are uniformly continuous. \( \alpha \) is real constant. Give conditions which guarantee that \( fg \) is uniformly continuous.

10. Let \( f : (X, d) \to (Y, \rho) \) be uniformly continuous. Show:

(a) If \((x_n)\) is a Cauchy sequence in \((X, d)\), then \((f(x_n))\) is a Cauchy sequence in \((Y, \rho)\).

(b) If \((X, d)\) is totally bounded, then \( f(X) \) is totally bounded in \((Y, \rho)\).

11. Show that \((M, d)\) is complete if and only if, for every \( r > 0 \), the closed ball \( \overline{B}_r(x) \) is complete.

12. Define \( d(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right| \) for \( n, m \in \mathbb{N} \). Show that \( d \) is a metric on \( \mathbb{N} \) and that \((\mathbb{N}, d)\) is not complete.

13. Show that the following are equivalent:

(a) \((X, d)\) is complete.
(b) If \((F_n)\) is a sequence of nonempty closed subsets of \(X\) such that \(F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots\) and \(\text{diam } F_n \to 0\), then \(\bigcap_{n \geq 1} F_n \neq \emptyset\). Here \(\text{diam } F_n\) is the diameter of the set \(F_n\), which is defined by \(\text{diam } F_n := \sup \{d(x, y) \mid x, y \in F_n\}\).

14. Let \((X, d)\) be a complete metric space and \(f : X \to X\). Define \(g(x) = f(f(x))\), \(x \in X\), that is, \(g = f \circ f\). Assume that the map \(g : X \to X\) is a contraction. Show that \(f\) has a unique fixed point.

15. Consider \(\mathbb{R}\) with the standard metric. Assume that \(f : \mathbb{R} \to \mathbb{R}\) is a contraction, \(|f(x) - f(y)| \leq \alpha |x - y|\) for some \(\alpha \in (0, 1)\) and all \(x, y \in \mathbb{R}\). Define \(F(x) = x + f(x)\). Show that \(F : \mathbb{R} \to \mathbb{R}\) is one-one, \(F(\mathbb{R}) = \mathbb{R}\), and that \(F\) and \(F^{-1}\) are continuous.

16. (a) Consider \(\mathbb{R}^2\) with the standard metrics. Which of the following subsets of \(\mathbb{R}^2\) are compact? \(A = \{(x, y) \mid x^2 + y^2 < 1\}\), \(B = \{(x, y) \mid |x| + |y| \leq 1\}\), \(C = \{(x, y) \mid x \geq 1, 0 \leq y \leq 1/x\}\).

(b) Consider \(\mathbb{Q}\) with the standard metric and let \(A = \{x \in \mathbb{Q} \mid 2 < x^2 < 3\}\). Show that \(A\) is closed and bounded but not compact.

17. Assume that \(A\) and \(B\) are subsets of a metric space \((X, d)\). Define the distance between \(A\) and \(B\) by \(d(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\}\).

(a) Show that if \(A\) is closed, \(B\) is compact, and \(A \cap B = \emptyset\), then \(d(A, B) > 0\).

(b) True or false: if \(A\) and \(B\) are closed and \(A \cap B = \emptyset\), then \(d(A, B) > 0\).

18. Let \(A\) be a subset of \(\mathbb{R}^n\) considered with the standard metric. Assume that every continuous function \(f : A \to \mathbb{R}\) is bounded. Show that \(A\) is compact.

19. Let \((X, d)\) be a compact metric space and suppose that \(f : X \to X\) satisfies \(d(f(x), f(y)) < d(x, y)\) whenever \(x \neq y\). Show that \(f\) has a fixed point. \(\text{Hint:}\) First note that \(f\) is continuous. Next consider the function \(g(x) = d(x, f(x))\).

20. Show that \((X, d)\) is compact if and only if whenever \((F_n)_{n \geq 1}\) is a sequence of nonempty closed subsets of \(X\) satisfying \(F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots\), then \(\bigcap_{n \geq 1} F_n \neq \emptyset\).